On the Lattice Isomorphism Problem, Quadratic Forms, Remarkable Lattices, and Cryptography

Léo Ducas, Wessel van Woerden (CWI, Cryptology Group).

Centrum Wiskunde & Informatica

Motivation

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- General identification, encryption and signature scheme based on the Lattice Isomorphism Problem.

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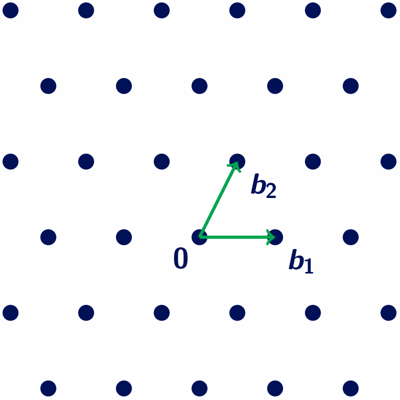
- Many wonderful lattices exist with great geometric properties.
- Can we use these in cryptography?

Contributions

- General identification, encryption and signature scheme based on the Lattice Isomorphism Problem.
- Better lattice $\implies$ better efficiency and security.

Lattices

Lattice $\mathcal{L}(\boldsymbol{B}) := \{\sum_i x_i \boldsymbol{b}_i : \boldsymbol{x} \in \mathbb{Z}^n\} \subset \mathbb{R}^n$



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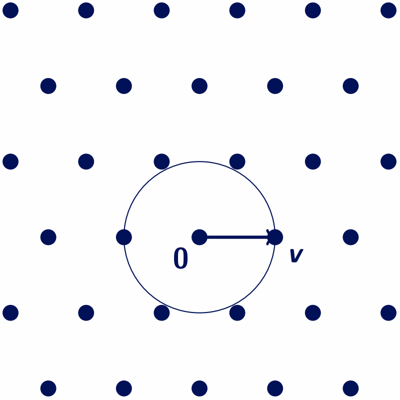

First minimum

$$\lambda_1(\mathcal{L}) := \min_{\boldsymbol{x} \in \mathcal{L} \setminus \{\boldsymbol{0}\}} \|\boldsymbol{x}\|_2$$

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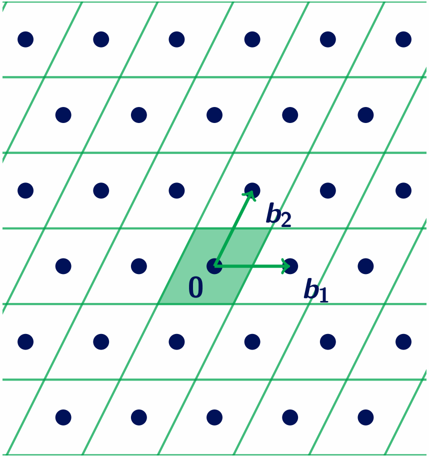

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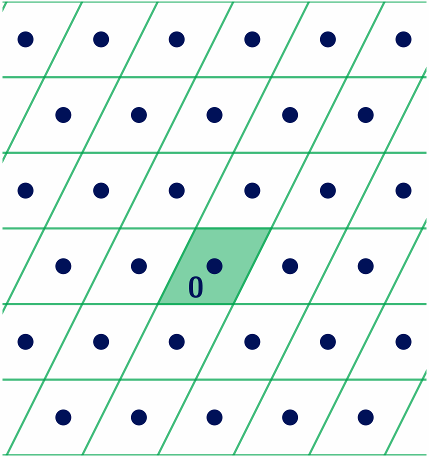

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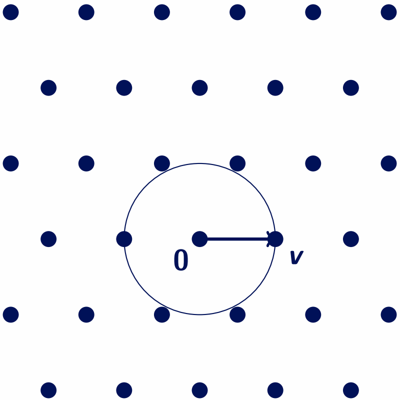
Minkowski's Theorem

$$\lambda_1(\mathcal{L}) \leq \underbrace{2 \frac{\det(\mathcal{L})^{1/n}}{\mathrm{vol}(\mathcal{B}^n)^{1/n}}}_{\mathrm{Mk}(\mathcal{L})} \leq \sqrt{n} \det(\mathcal{L})^{1/n}$$

Lattice $\mathcal{L} \subset \mathbb{R}^n$



SVP

Find a *shortest* nonzero vector $\boldsymbol{v} \in \mathcal{L}$ of length $\lambda_1(\mathcal{L}) \leq \mathsf{Mk}(\mathcal{L})$.

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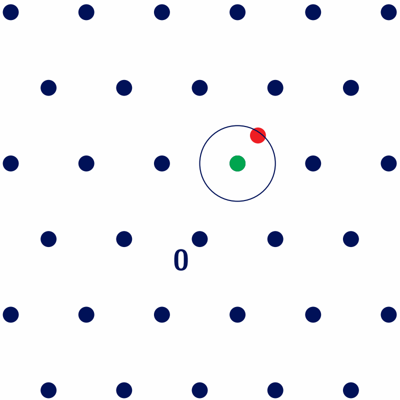

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BDD

Given a target $\boldsymbol{t} = \boldsymbol{v} + \boldsymbol{e} \in \mathbb{R}^n$ with $\boldsymbol{v} \in \mathcal{L}$ and $\|\boldsymbol{e}\| < \rho \leq \frac{1}{2}\lambda_1(\mathcal{L}) \leq \frac{1}{2}\mathsf{Mk}(\mathcal{L})$,

recover $\boldsymbol{v} \in \mathcal{L}$.

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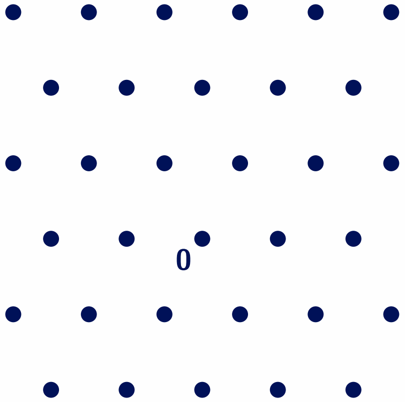

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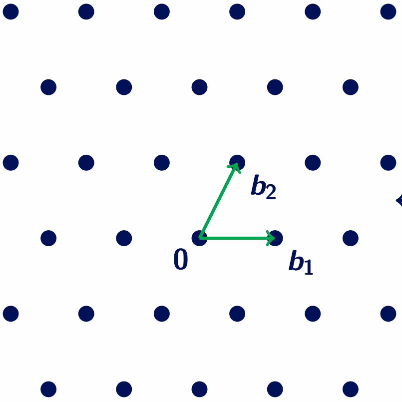
Hardness depends on the *gaps*!

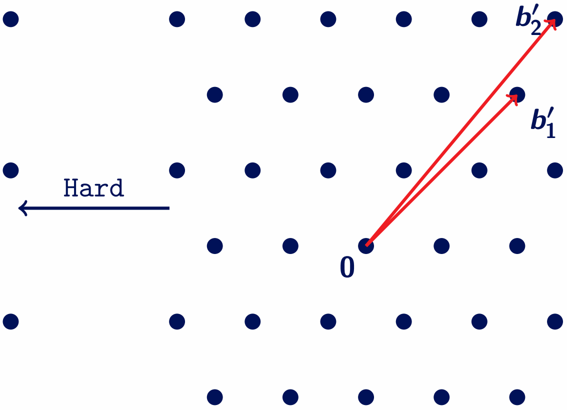
Encryption, legacy approach

Good basis (Secret key)



$\boldsymbol{B}$

Bad basis (Public key)



$\boldsymbol{B}' = \boldsymbol{B} \cdot \boldsymbol{U}, \ \boldsymbol{U} \in \mathrm{GL}_d(\mathbb{Z})$

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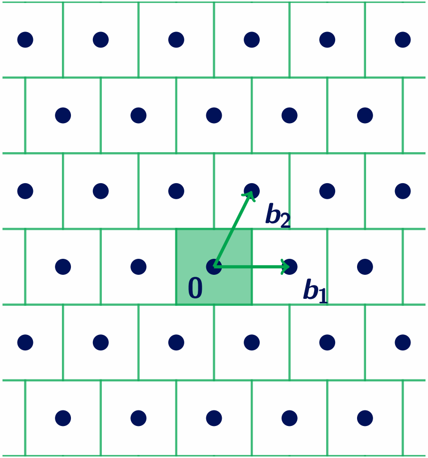

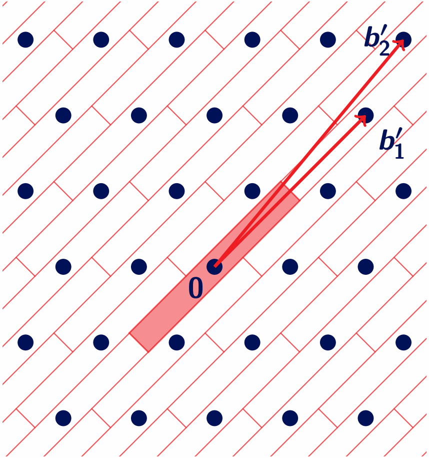

Babai's nearest plane algorithm

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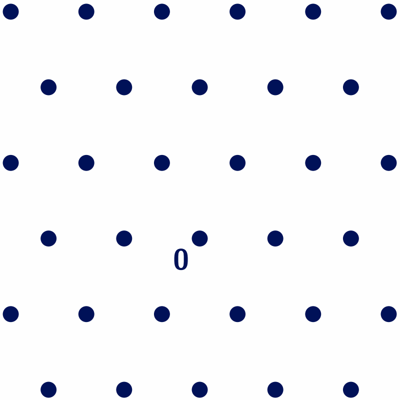

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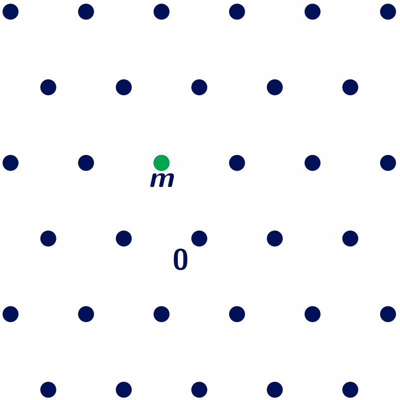

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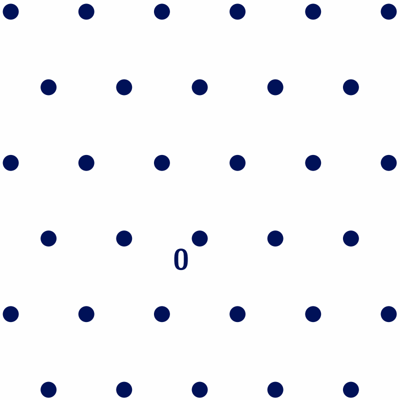

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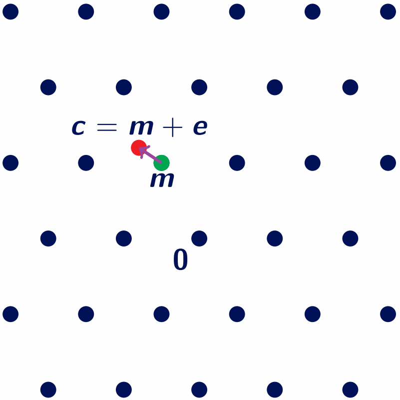
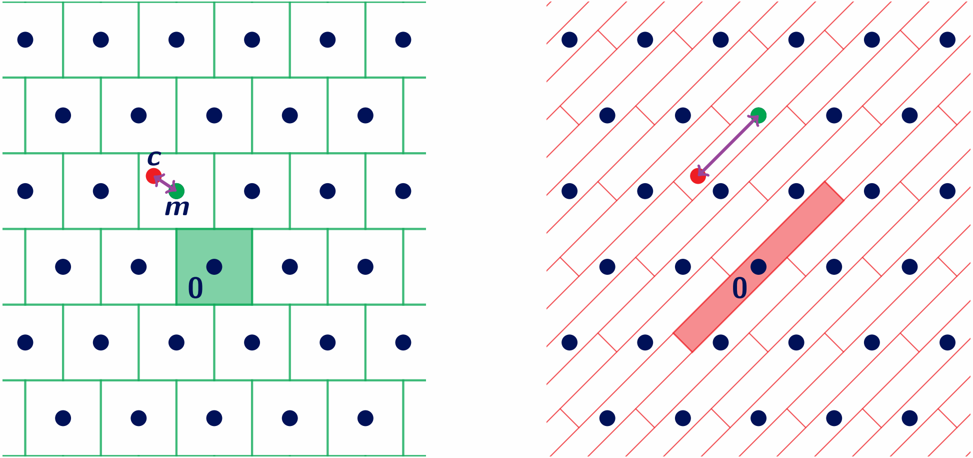


Encrypt by adding a small error

Encryption, legacy approach

Good basis (Secret key) | Bad basis (Public key)



Decrypt using the good basis

Large gap

Current lattice based crypto relies on hardness of decoding with

$$\mathrm{gap}(\mathcal{L}, \rho) \geq \Omega(\sqrt{n}).$$

Broken by SVP in dimension $\beta \leq n/2 + o(n)$, e.g. $n = 1024 \implies \beta \approx 450$.

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An example: Prime Lattice [CR88]

Let $p_1, \ldots, p_n$ be distinct small primes not dividing m, we define:

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Remarkable Lattices

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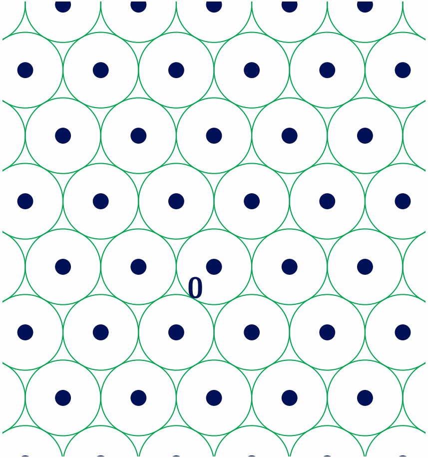
- Efficiently decode up to large radius ρ by trial division.
- With the right parameters $\mathrm{gap}(\mathcal{L}_{\mathtt{prime}}, \rho) = \Theta(\log(n))$ [DP19].

How to hide the remarkable lattice?

Good lattice (~~Secret key~~)



$\mathcal{L}$

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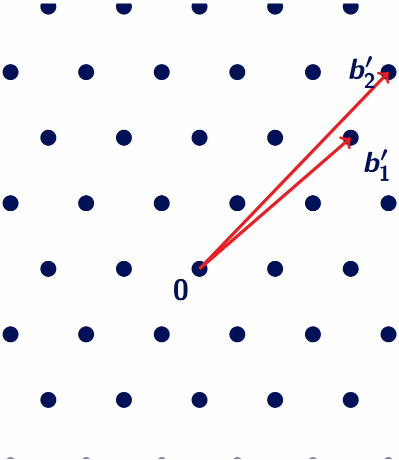

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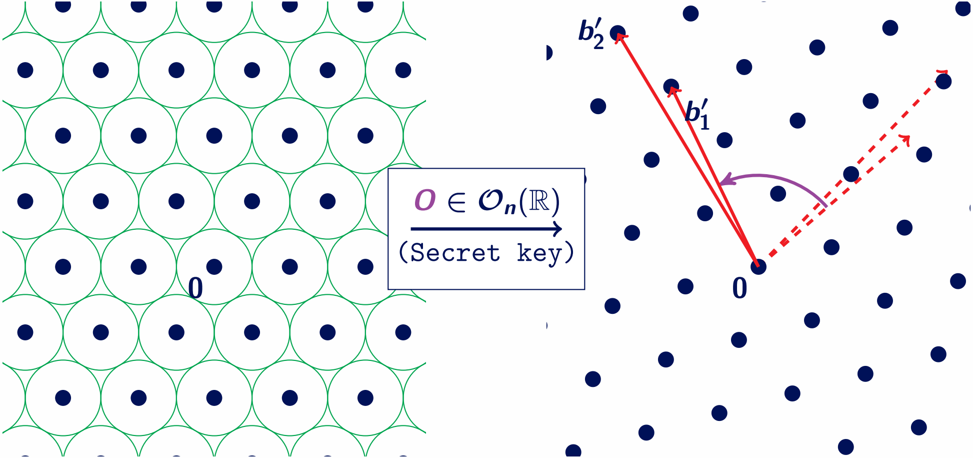

$\mathcal{L}$ $O \cdot \mathcal{L}$

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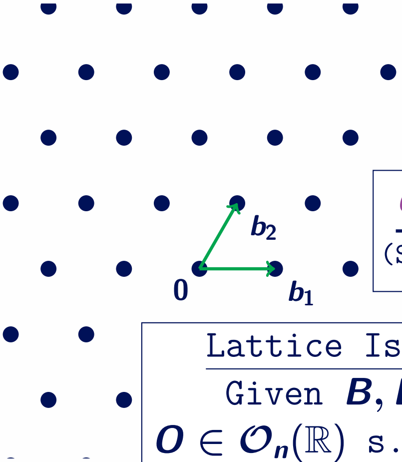
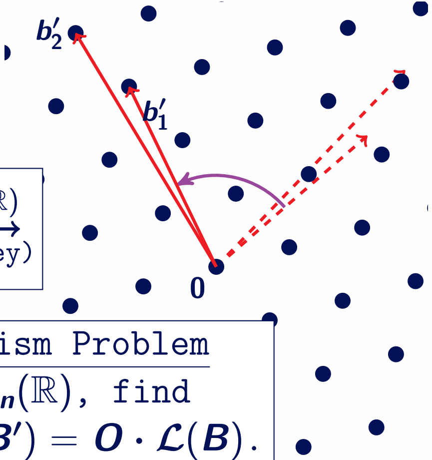


$$\xrightarrow[\text{(Secret key)}]{O \in \mathcal{O}_n(\mathbb{R})}$$



Lattice Isomorphism Problem

Given $B, B' \in \mathrm{GL}_n(\mathbb{R})$, find $O \in \mathcal{O}_n(\mathbb{R})$ s.t. $\mathcal{L}(B') = O \cdot \mathcal{L}(B)$.

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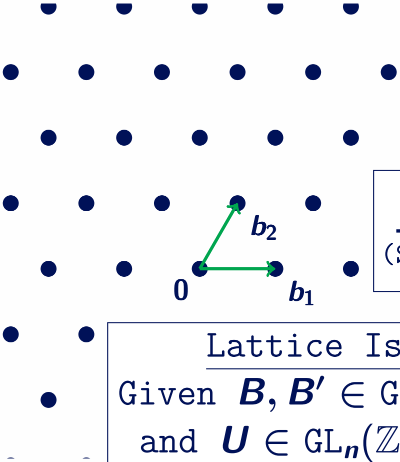

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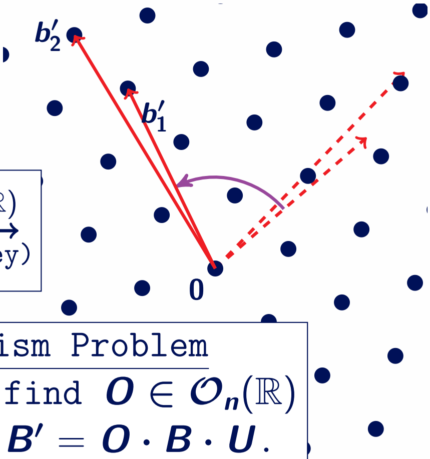

$B' = O \cdot B \cdot U$

Lattice Isomorphism Problem

Given $B, B' \in \mathrm{GL}_n(\mathbb{R})$, find $O \in \mathcal{O}_n(\mathbb{R})$ and $U \in \mathrm{GL}_n(\mathbb{Z})$ s.t. $B' = O \cdot B \cdot U$.

Lattice Isomorphism Problem

LIP

Given $\boldsymbol{B}, \boldsymbol{B}' \in \mathrm{GL}_n(\mathbb{R})$ of isomorphic lattices, find $\boldsymbol{O} \in \mathcal{O}_n(\mathbb{R})$ and $\boldsymbol{U} \in \mathrm{GL}_n(\mathbb{Z})$ s.t. $\boldsymbol{B}' = \boldsymbol{O} \cdot \boldsymbol{B} \cdot \boldsymbol{U}$.

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- The lattice analogue of 'vintage' McEliece $\mathbf{G}' = \mathbf{P} \cdot \mathbf{G} \cdot \mathbf{S}$,
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- The lattice analogue of 'vintage' McEliece $G' = P \cdot G \cdot S$,
- and Oil and Vinegar $\mathcal{P} = \mathcal{Q} \circ \mathcal{S}$.
- Best known attacks require to solve SVP.

Algorithms

- $\mathrm{Min}(\mathcal{L}(B')) = O \cdot \mathrm{Min}(\mathcal{L}(B))$.
- Best practical algorithm: backtrack search all isometries between the sets of short vectors.
- Best proven algorithm uses short primal and dual vectors ($n^{O(n)}$ time and space).

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Sidestep real values!

$\boldsymbol{O} \in \mathcal{O}_{\boldsymbol{n}}(\mathbb{R})$

Sample $\boldsymbol{U} \in \mathrm{GL}_{\boldsymbol{n}}(\mathbb{Z})$ s.t. $\boldsymbol{B}'$ is independent of $\boldsymbol{B}$.

$$\boldsymbol{B}' = \boldsymbol{O} \cdot \boldsymbol{B} \cdot \boldsymbol{U}.$$

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$$Q := B^t B \in \mathcal{S}_n^{>0}$$

Lattices $\Longrightarrow$ Quadratic Forms

$$(\mathcal{L} \subset \mathbb{R}^n, \langle x, y \rangle) \Longrightarrow (\mathbb{Z}^n, \langle x, y \rangle_Q := x^t Q y)$$

Keep the geometry, forget the embedding.

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LIP restated:

Find $U \in \mathrm{GL}_n(\mathbb{Z})$ s.t. $Q' = U^t Q U$.

Unimodular $\boldsymbol{U} \in \mathrm{GL}_{\boldsymbol{n}}(\mathbb{Z})$

An average-case distribution

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Equivalence class $[\boldsymbol{Q}] := \{\boldsymbol{U}^t \boldsymbol{Q} \boldsymbol{U} : \boldsymbol{U} \in \mathrm{GL}_{\boldsymbol{n}}(\mathbb{Z})\}$.

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$$+ \quad \boxed{\begin{array}{c}\text{Efficient sampler } (\boldsymbol{Q}', \boldsymbol{U}) \leftarrow \mathrm{Sample}_\sigma(\boldsymbol{Q}) \\ \text{s.t. } \quad \boldsymbol{Q}' \sim \mathcal{D}_\sigma([\boldsymbol{Q}]) \text{ and } \boldsymbol{Q}' = \boldsymbol{U}^t \boldsymbol{Q} \boldsymbol{U}.\end{array}}$$

$\boldsymbol{Q}'$ only depends on the class $[\boldsymbol{Q}]$ and not on $\boldsymbol{Q}$ itself.

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$\implies$ average-case LIP, ZKPoK and identification scheme.

$\implies$ Worst-case to average-case reduction over $[\boldsymbol{Q}]$.

An average-case distribution

- ac-$\mathrm{LIP}_{\sigma}^{\boldsymbol{Q}}$: given $\boldsymbol{Q}$ and $\boldsymbol{Q}' \leftarrow \mathcal{D}_\sigma([\boldsymbol{Q}])$, recover $\boldsymbol{U} \in \mathrm{GL}_{\boldsymbol{n}}(\mathbb{Z})$.

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- ac-LIP$^{\boldsymbol{Q}}_{\sigma}$: given $\boldsymbol{Q}$ and $\boldsymbol{Q}' \leftarrow \mathcal{D}_\sigma([\boldsymbol{Q}])$, recover $\boldsymbol{U} \in \mathrm{GL}_n(\mathbb{Z})$.
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$$\begin{tikzcd}
\boldsymbol{Q}_0 \arrow[r, "\boldsymbol{U}"] \arrow[d, "\boldsymbol{V}"'] & \boldsymbol{Q}_1 \arrow[dl, dashed, "\boldsymbol{U}^{-1}\boldsymbol{V}"] \\
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\end{tikzcd}$$

$$(\boldsymbol{Q}', \boldsymbol{V}) \leftarrow \mathtt{Sample}_\sigma(\boldsymbol{Q}_0)$$

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$$(\boldsymbol{Q}', \boldsymbol{V}) \leftarrow \mathrm{Sample}_\sigma(\boldsymbol{Q}_0)$$

- Worst-case to average-case reduction:

$$\begin{tikzcd} \boldsymbol{Q} \arrow[r, leftrightarrow, "\mathrm{WC}"] \arrow[dr, leftrightarrow, dashed, "\mathrm{AC}"'] & \boldsymbol{Q}' \arrow[d, "\boldsymbol{U}'"] \\ & \boldsymbol{Q}'' \end{tikzcd}$$

$$(\boldsymbol{Q}'', \boldsymbol{U}') \leftarrow \mathrm{Sample}_\sigma(\boldsymbol{Q}')$$

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Decoding attack (BDD): $\mathsf{gap}(\mathcal{L}, \rho)$.

Decodable Lattices

Lattice	$\mathrm{gap}(\mathcal{L})$	$\mathrm{gap}(\mathcal{L}^*)$	$\mathrm{gap}(\mathcal{L}, \rho)$
'Random' Lattice	$\Theta(1)$	$\Theta(1)$	$2^{\Theta(n)}$
Prime Lattice	$\Theta(\log n)$	$\Omega(\sqrt{n})$	$\Theta(\log n)$ [CR88, DP19]
Barnes-Sloane	$\Theta(\sqrt{\log n})$	$\Omega(\sqrt{n})$	$\Theta(\sqrt{\log n})$ [MP20]
Reed-Solomon	$\Theta(\sqrt{\log n})$	$\Omega(\sqrt{n})$	$\Theta(\sqrt{\log n})$ [BP22]
$\mathbb{Z}^n$	$\Theta(\sqrt{n})$	$\Theta(\sqrt{n})$	$\Theta(\sqrt{n})$
NTRU, LWE	$\Omega(1) \ldots O(\sqrt{n})$	$\Omega(1)$	$\Omega(\sqrt{n})$
Barnes-Wall	$\Theta(\sqrt[4]{n})$	$\Theta(\sqrt[4]{n})$	$\Theta(\sqrt[4]{n})$ [MN08]

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  - Tune parameters based on concrete cryptanalysis.

HAWK - Performance

	Falcon-512	Hawk-512	$\left(\frac{\text{Falcon}}{\text{Hawk}}\right)$
AVX2 KeyGen	**8.10** ms	**4.13** ms	$\times$**1.96**
Reference KeyGen	**18.76** ms	**13.78** ms	$\times$**1.36**
AVX2 Sign	**200** µs	**47** µs	$\times$**4.3**
Reference Sign	**2401** µs	**206** µs	$\times$**11.7**
AVX2 Verify	**51** µs	**20** µs	$\times$**2.6**
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Secret key (bytes)	**1281**	**1153**	$\times$**1.11**
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	Uncompressed HAWK-512
Reference Sign	**185** µs
Reference Verify	**238** µs
Signature (bytes)	**1223** $\pm$ **7**

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Thanks! :)