On the Lattice Isomorphism Problem, Quadratic Forms, Remarkable Lattices, and Cryptography

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Centrum Wiskunde & Informatica

Motivation

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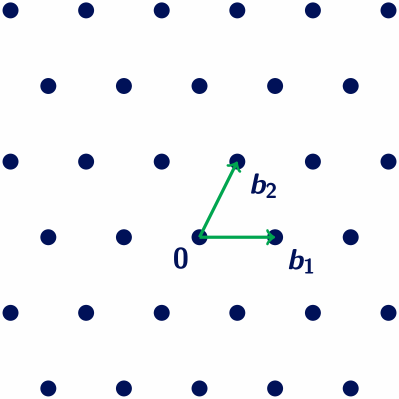
- General identification, encryption and signature scheme based on the Lattice Isomorphism Problem.
- Better lattices $\implies$ better efficiency and security.
- Lots of open questions.

Lattice $\mathcal{L}(\boldsymbol{B}) := \{\sum_i x_i \boldsymbol{b}_i : \boldsymbol{x} \in \mathbb{Z}^n\} \subset \mathbb{R}^n$



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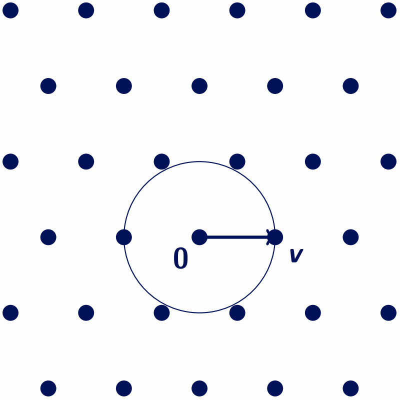

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$$\lambda_1(\mathcal{L}) := \min_{x \in \mathcal{L} \setminus \{0\}} \|x\|_2$$

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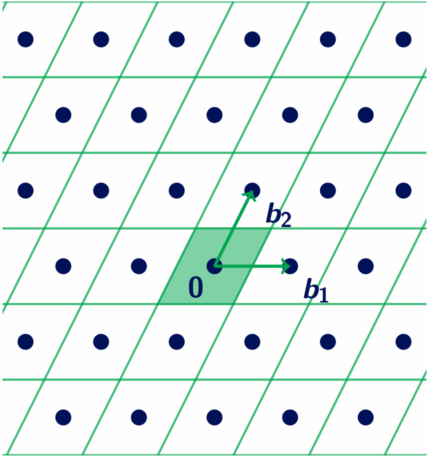

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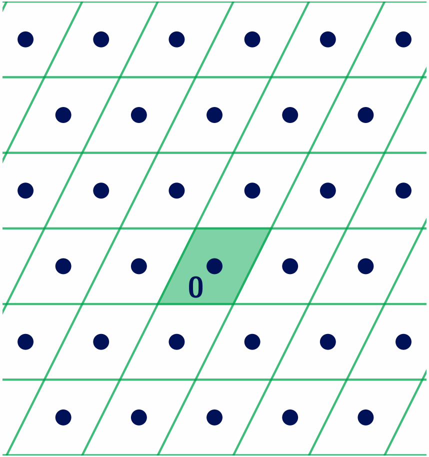

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Minkowski's Theorem

$$\lambda_1(\mathcal{L}) \leq \underbrace{2 \frac{\det(\mathcal{L})^{1/n}}{\mathrm{vol}(\mathcal{B}^n)^{1/n}}}_{\mathrm{Mk}(\mathcal{L})} \leq \sqrt{n} \det(\mathcal{L})^{1/n}$$

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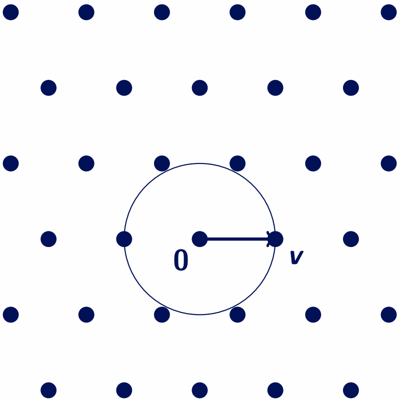

SVP

Find a *shortest* nonzero vector $\boldsymbol{v} \in \mathcal{L}$ of length $\lambda_1(\mathcal{L}) \leq \mathsf{Mk}(\mathcal{L})$.

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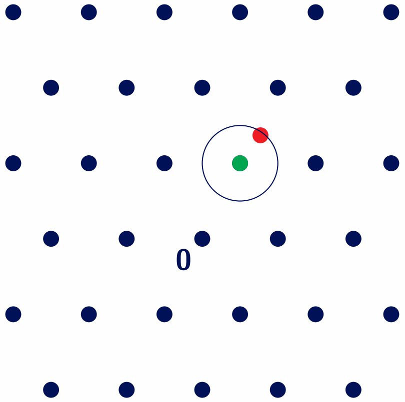

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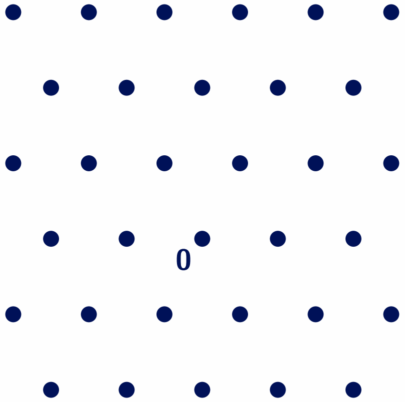

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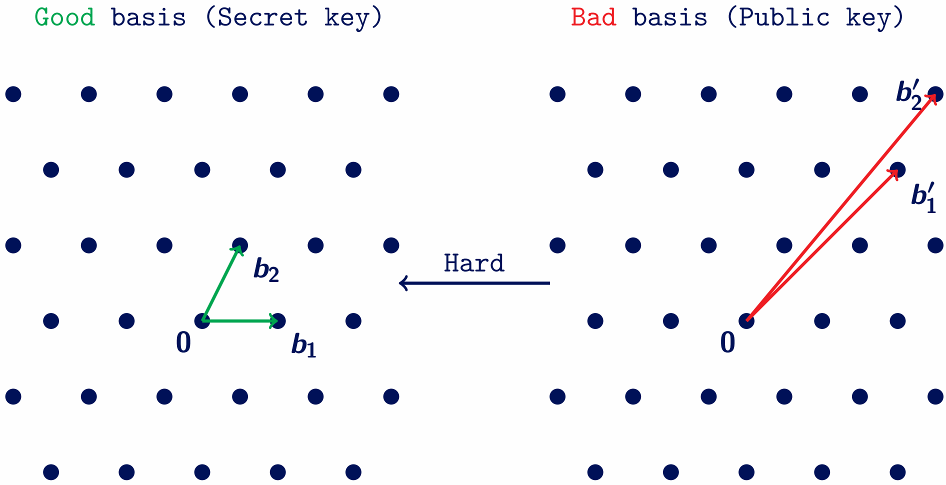
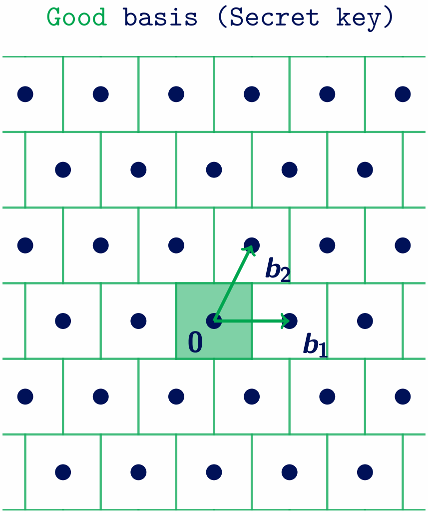
Hardness depends on the *gap*
$\mathsf{gap}(\mathcal{L}) := \frac{\mathsf{Mk}(\mathcal{L})}{\lambda_1(\mathcal{L})}$ or $\mathsf{gap}(\mathcal{L}, \rho) := \frac{\mathsf{Mk}(\mathcal{L})}{\rho}$.
(state-of-art heuristic algorithms)
[ADPS16], [AGVW17], [PV21]

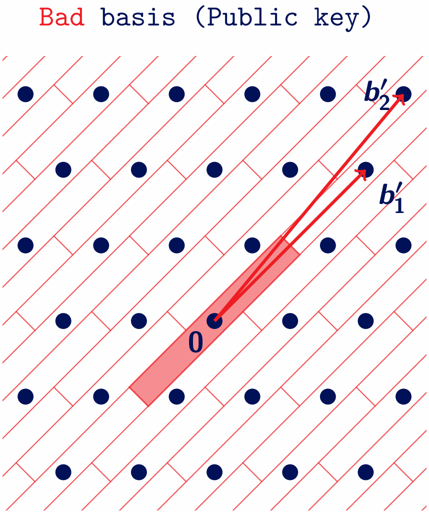
How to do encryption?



$\boldsymbol{B}$

$\boldsymbol{B}' = \boldsymbol{B} \cdot \boldsymbol{U},\ \ \boldsymbol{U} \in \mathrm{GL}_d(\mathbb{Z})$

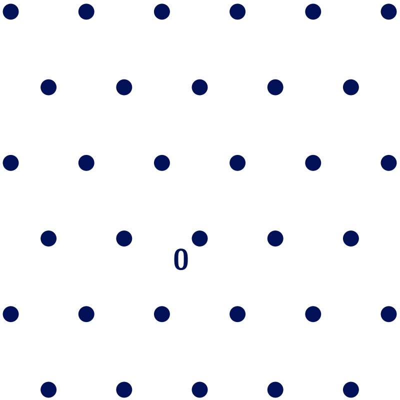




Babai's nearest plane algorithm

How to do encryption?

Good basis (Secret key)



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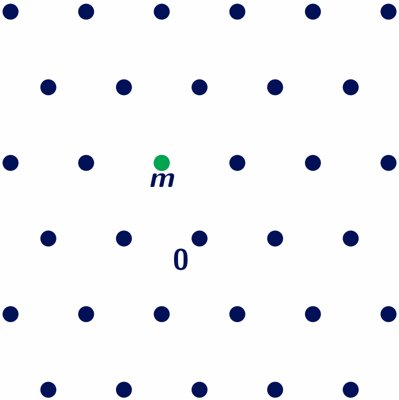

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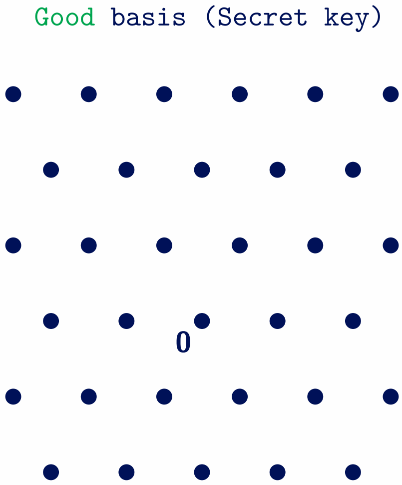
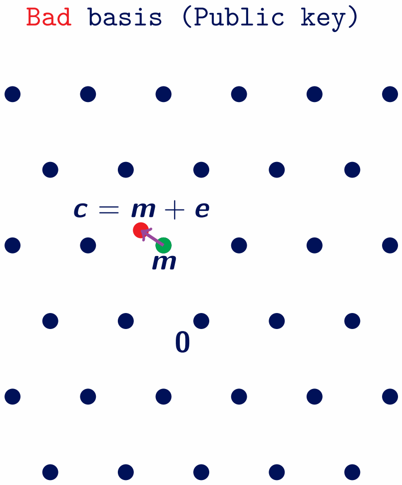
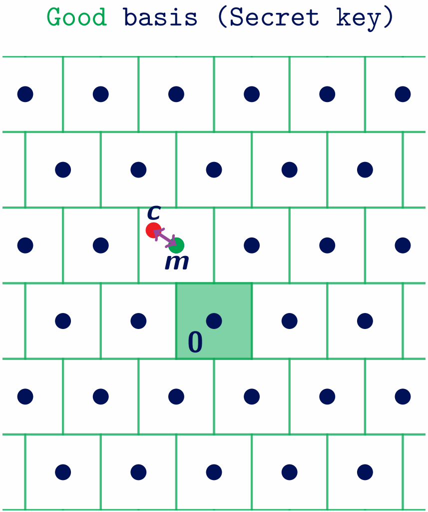
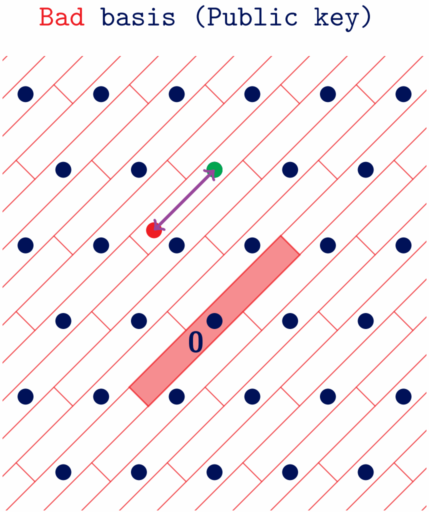




Encrypt by adding a small error

How to do encryption?





Decrypt using the good basis

Large gap

Current lattice based crypto relies on hardness of decoding with

$$\mathrm{gap}(\mathcal{L}, \rho) \geq \Omega(\sqrt{n}).$$

Broken by SVP in dimension $\beta \leq n/2 + o(n)$.

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An example: Prime Lattice [CR88]

Let $p_1, \ldots, p_n$ be distinct small primes not dividing m, we define:

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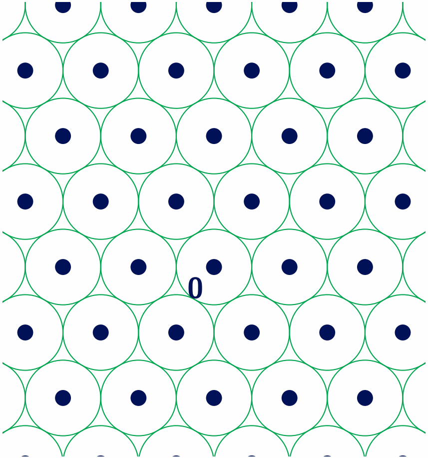
- Efficiently decode up to large radius ρ by trial division.
- With the right parameters $\mathrm{gap}(\mathcal{L}_{\mathtt{prime}}, \rho) = \Theta(\log(n))$ [DP19].

How to hide the remarkable lattice?

Good lattice (~~Secret key~~)



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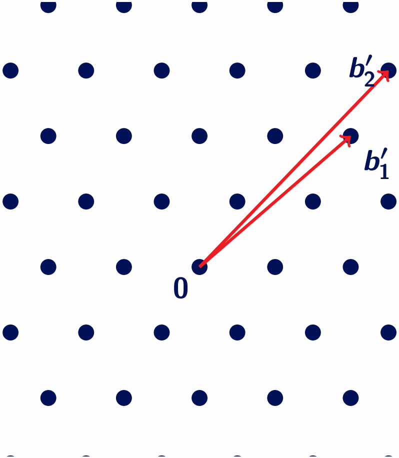

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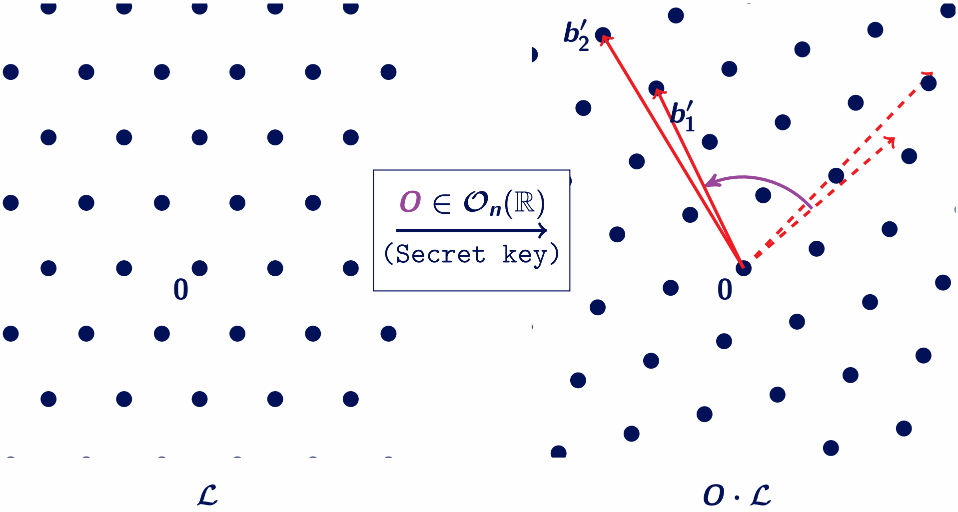

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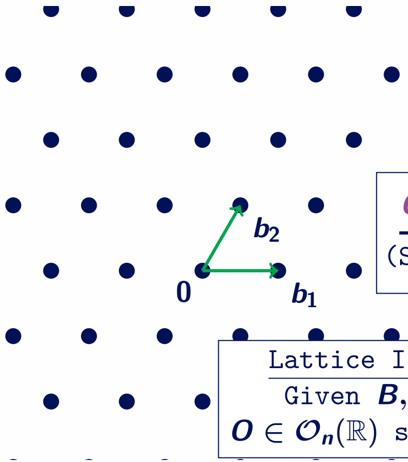

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$O \in \mathcal{O}_n(\mathbb{R})$ ⟶ (Secret key)

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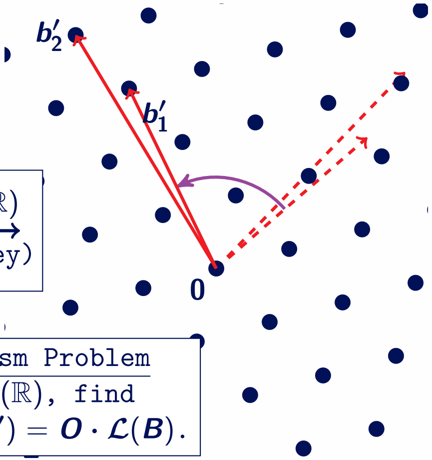

$O \cdot \mathcal{L}$

Lattice Isomorphism Problem

Given $B, B' \in \mathrm{GL}_n(\mathbb{R})$, find $O \in \mathcal{O}_n(\mathbb{R})$ s.t. $\mathcal{L}(B') = O \cdot \mathcal{L}(B)$.

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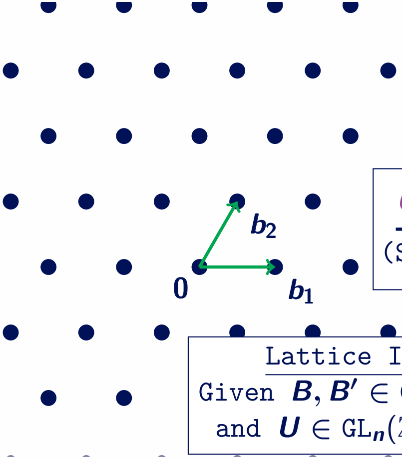

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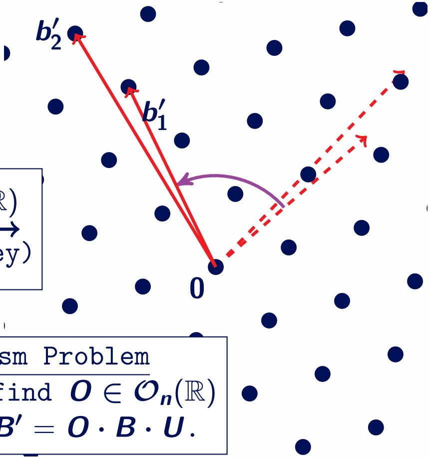

$B' = O \cdot B \cdot U$

$\xrightarrow[\text{(Secret key)}]{O \in \mathcal{O}_n(\mathbb{R})}$

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Given isomorphic $\boldsymbol{B}, \boldsymbol{B}' \in \mathrm{GL}_{\boldsymbol{n}}(\mathbb{R})$, find $\boldsymbol{O} \in \mathcal{O}_{\boldsymbol{n}}(\mathbb{R})$ and $\boldsymbol{U} \in \mathrm{GL}_{\boldsymbol{n}}(\mathbb{Z})$ s.t. $\boldsymbol{B}' = \boldsymbol{O} \cdot \boldsymbol{B} \cdot \boldsymbol{U}$.

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Algorithms

- $\mathrm{Min}(\mathcal{L}(\mathbf{B}')) = \mathbf{O} \cdot \mathrm{Min}(\mathcal{L}(\mathbf{B}))$.
- Best practical algorithm: backtrack search all isometries between the sets of short vectors.
- Best proven algorithm uses short primal and dual vectors ($n^{O(n)}$ time and space).

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LIP (restated)

Given equivalent $Q, Q' \in \mathcal{S}_n^{>0}(\mathbb{R})$, find $U \in \mathrm{GL}_n(\mathbb{Z})$ s.t. $Q' = U^t Q U$.

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- Only work with $Q \in \mathcal{S}_n^{>0}(\mathbb{Z})$.

Encryption [informal]

Prerequisite

Let S be a quadratic form with an efficient decoder up to some radius $\rho < \lambda_1(S)/2$.

Keygen :

Sample $(pk, sk) := (P, U) \leftarrow \mathcal{D}_\sigma([S])$, such that $P = U^t S U$.

Encrypt(P, m) :

$$c := m + e \text{ s.t. } \|e\|_P \leq \rho$$

Decrypt(U, c) :

$$m' := \mathrm{Decode}(S, Uc) \text{ s.t. } \|m' - Uc\|_S \leq \rho$$

$$m = U^{-1} m'$$

Average case instances

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2. Let $Y = UT$ be the unique upper triangular HNF decomposition.
3. Return $(R = U^t S U, U)$.

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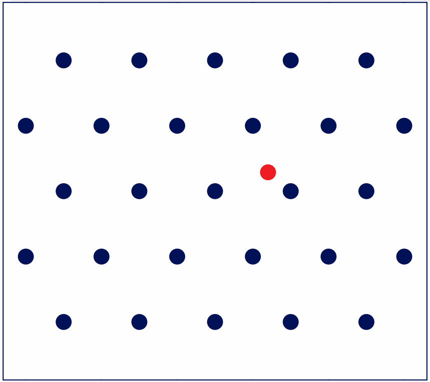
Properties

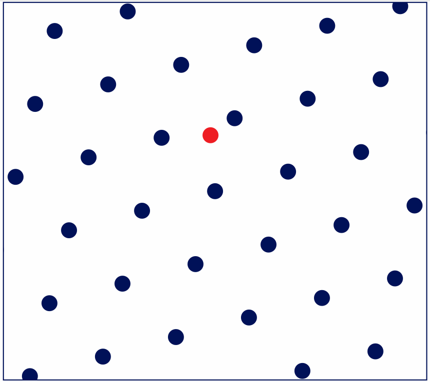
- $\mathbf{R}$ only depends on the class $[\mathbf{Q}]$ and σ (ZKPoK, identification).
- Defines an average-case LIP problem ac-$\mathrm{LIP}_\sigma^{\mathbf{S}}$.
- Given any representative we can sample at $\sigma \geq 2^{\Theta(n)} \cdot \lambda_n([\mathbf{S}])$ ($\implies$ worst-case to average-case reduction).

Actual hardness assumption

1. For a uniformly random $O \in \mathcal{O}_n(\mathbb{R})$, decoding in $O \cdot \mathcal{L}_0$ is hard.



$\mathcal{L}_0$

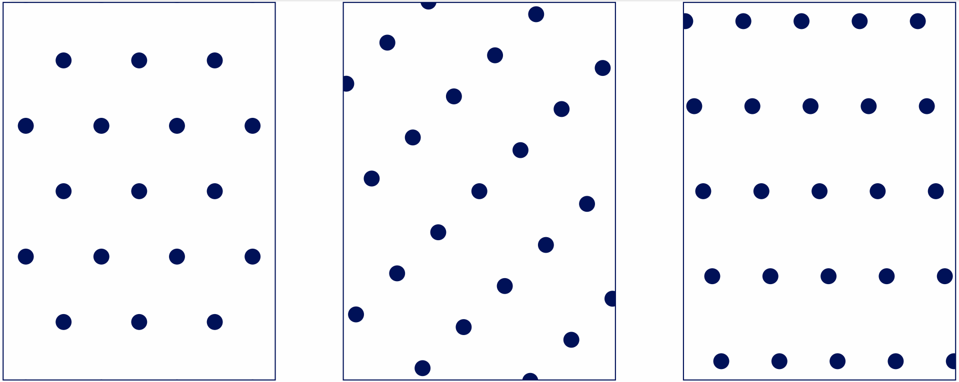


$\mathcal{L}_1 = O \cdot \mathcal{L}_0$

Distinguishing LIP

$\Delta\,\mathsf{LIP}_{\sigma}^{Q_0,Q_1}$

Given two quadratic forms $Q_0, Q_1 \in \mathcal{S}_n^{>0}$, and $Q \in \mathcal{D}_\sigma([Q_b])$ for a uniform random $b \in \{0,1\}$, find b.

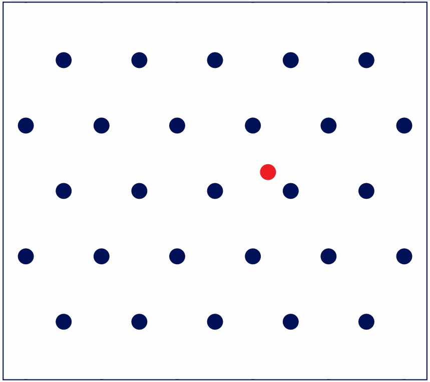


$\mathcal{L}_0$ $\qquad$ $O \cdot \mathcal{L}_b$ $\qquad$ $\mathcal{L}_1$

Security Proof

Security Assumption [informal]

1. $\mathcal{O} \cdot \mathcal{L}_0$ is *indistinguishable* from a *random* lattice.
2. Decoding in a *random* lattice is hard.



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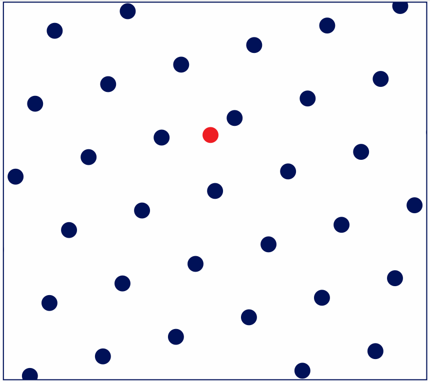
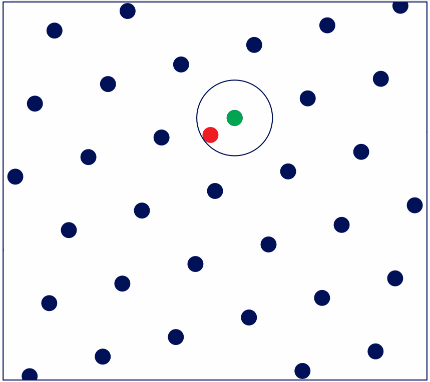


$\mathcal{L}_1 = \mathcal{O} \cdot \mathcal{L}_0$

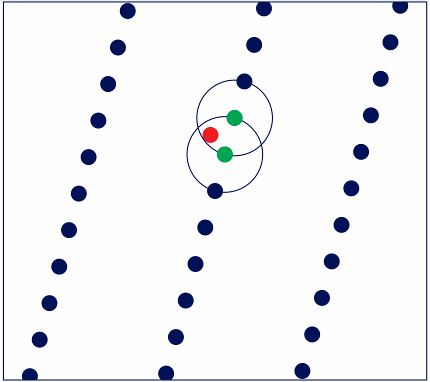
Security Assumption [informal]

1. ..*indistinguishable* from *some* lattice with a *dense* sublattice.

~~2. Decoding in a random lattice is hard.~~



$O \cdot \mathcal{L}_0$



$\mathcal{L}_1$

Arithmetic Invariants

- $\det(\boldsymbol{Q})$.
- $\gcd(\boldsymbol{Q}) := \gcd(\boldsymbol{Q}_{ij})_{i,j}$
- $\gcd\{\|\boldsymbol{x}\|_{\boldsymbol{Q}}^2 : \boldsymbol{x} \in \mathbb{Z}^n\}$
- Self dual? (up to scaling)

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- Equivalence over $\boldsymbol{R} \supset \mathbb{Z}$, $\boldsymbol{U} \in \mathrm{GL}_n(\boldsymbol{R})$, $\boldsymbol{R} \in \{\mathbb{R}, \mathbb{Q}, \forall \boldsymbol{p}\ \mathbb{Q}_{\boldsymbol{p}}, \forall \boldsymbol{p}\ \mathbb{Z}_{\boldsymbol{p}}\}$

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The Genus of $Q \in \mathcal{S}_n^{>0}(\mathbb{Z})$ represents the $\mathbb{Z}_p$-equivalence classes $[Q]_{\mathbb{Z}_p}$ for $p = 2$ and all primes $p | \det(Q)$.

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Genus attack

If $\mathrm{genus}(\boldsymbol{Q}_0) \neq \mathrm{genus}(\boldsymbol{Q}_1)$, then $\Delta\,\mathrm{LIP}^{\boldsymbol{Q}_0,\boldsymbol{Q}_1}$ is easy.

- If the genera match, we have to distinguish by geometric invariants.

SVP Attack

If $\lambda_1(Q_0) \neq \lambda_1(Q_1)$, then $\Delta\,\mathsf{LIP}^{Q_0,Q_1} \leq \mathsf{SVP}$, with Minkowski Gap $\max\{\mathrm{gap}(Q_0), \mathrm{gap}(Q_1)\}$.

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Open Question

Are there better attacks when the genera match?

Instantiating (simple)

Theorem [informal]

Let $\mathcal{L}_0$ be a decodable lattice, and let $\mathcal{L}_1$ be a lattice with a dense sublattice, then our scheme is CPA-secure if $\Delta\,\mathsf{LIP}^{Q_0,Q_1}$ is hard.

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- Let $\mathcal{L} \subset \mathbb{R}^{n/2}$ be a ρ-decodable lattice with integral gram matrix.
- For some $g \in \mathbb{Z}_{\geq 1}$ we define

$$\mathcal{L}_0 := g\mathcal{L} \oplus (g+1)\mathcal{L} \qquad \& \qquad \mathcal{L}_1 := \mathcal{L} \oplus g(g+1)\mathcal{L}.$$

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Cryptanalysis

Invariants: $\mathrm{genus}(\mathcal{L}_0) = \mathrm{genus}(\mathcal{L}_1)$.

SVP: if $\mathrm{gap}(\mathcal{L}) \leq f$, $\mathrm{gap}(\mathcal{L}^*) \leq f^*$ and $\mathrm{gap}(\mathcal{L}, \rho) \leq f'$, then

$$\max\{\mathrm{gap}(\mathcal{L}_0), \mathrm{gap}(\mathcal{L}_0^*), \mathrm{gap}(\mathcal{L}_1), \mathrm{gap}(\mathcal{L}_1^*)\} \leq O(\max(f, f^*) \cdot f^* \cdot f')$$

Decodable Lattices

Lattice	$f := \mathrm{gap}(\mathcal{L})$	$f^* := \mathrm{gap}(\mathcal{L}^*)$	$f' := \mathrm{gap}(\mathcal{L}, \rho)$
$\mathbb{Z}^n$	$\Theta(\sqrt{n})$	$\Theta(\sqrt{n})$	$\Theta(\sqrt{n})$
'Random' Lattice	$\Theta(1)$	$\Theta(1)$	$2^{\Theta(n)}$
NTRU, LWE, ⋯	$\Theta(1)$	$\Theta(1)$	$\Omega(\sqrt{n})$
Prime Lattice	$\Theta(\log n)$	$\Omega(\sqrt{n})$	$\Theta(\log n)$ [CR88, DP19]
Barnes-Sloane	$\Theta(\sqrt{\log n})$	$\Omega(\sqrt{n})$	$\Theta(\sqrt{\log n})$ [MP20]
Reed-Solomon	$\Theta(\sqrt{\log n})$	$\Omega(\sqrt{n})$	$\Theta(\sqrt{\log n})$ [BP22]
Barnes-Wall	$\Theta(\sqrt[4]{n})$	$\Theta(\sqrt[4]{n})$	$\Theta(\sqrt[4]{n})$ [MN08]

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Open Question

Can we construct a decodable lattice with $\max\{f, f^*, f'\} \leq \mathrm{polylog}(n)$?

Future work

Remarkable Lattices

Can we construct a decodable lattice with $\max\{f, f^*, f'\} \leq \text{polylog}(n)$?

LIP to Δ LIP?

Can we reduce the search version of LIP to the distinguishing version? (for $\mathbb{Z}^n$ we can [Szydlo03])

Genus Sampling

Can we sample 'random' $[Q']$ such that $\text{genus}(Q') = \text{genus}(Q)$. Is $[Q']$ expected to have a good geometry? Is decoding in $[Q']$ hard?

Module-LIP

LIP is easy for some Ideal lattices [Gentry-Szydlo, Lenstra-Silverberg]. Is rank $k \geq 2$ module-LIP secure?

Thank you! :)

Full paper at eprint.iacr.org/2021/1332