Lattice cryptography and cryptanalysis

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Plan

Part I

Lattice theory

- Lattices
- Hard problems

Cryptography

- Trapdoor bases
- Encryption, Signature

Cryptanalysis

- Lattice Sieving
- Basis Reduction

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Lattices used in cryptography

- SIS, LWE, decLWE
- Security proofs

Hardness Reductions

- search to decision
- WC to AC reductions

Algebraic Lattices

- Ideal and module lattices
- NTRU, RLWE, mod-LWE

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acknowledgements: many slides adapted from Alice Pellet-Mary!

https://apelletm.pages.math.cnrs.fr/page-perso/research.html

Lattice theory

From codes to lattices

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most important:

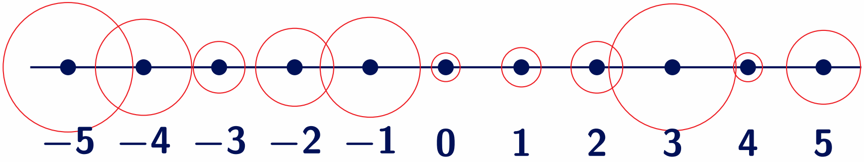
row vectors ($\boldsymbol{xG}$) $\longrightarrow$ column vectors ($\boldsymbol{Gx}$)

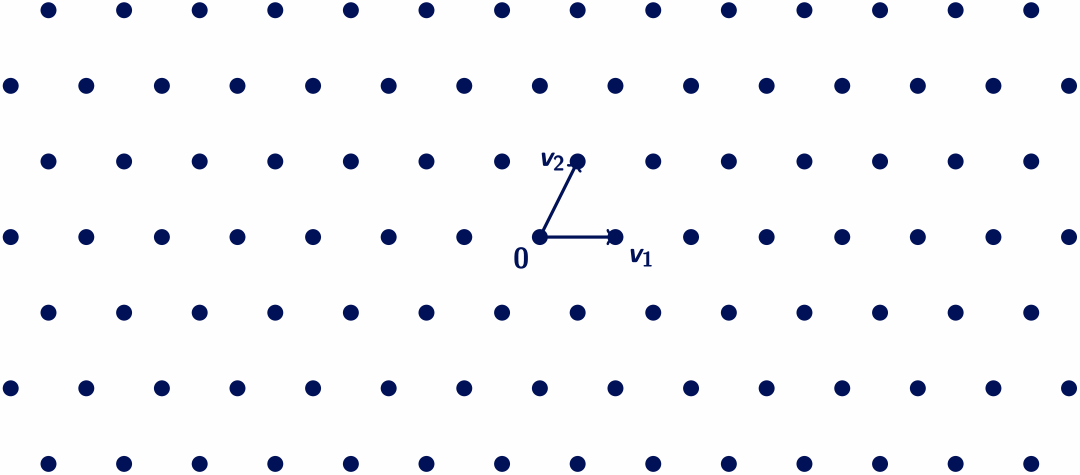
A lattice $\mathcal{L} \subset \mathbb{R}^n$ is a discrete subgroup of $\mathbb{R}^n$.

Discrete

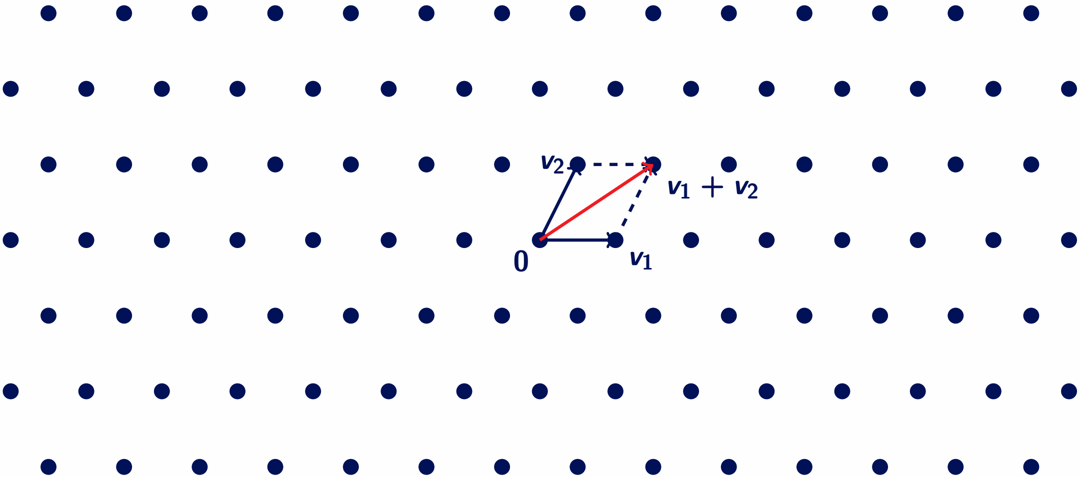
For every $v \in \mathcal{L}$ there exists an open ball around v that contains no other elements from $\mathcal{L}$.

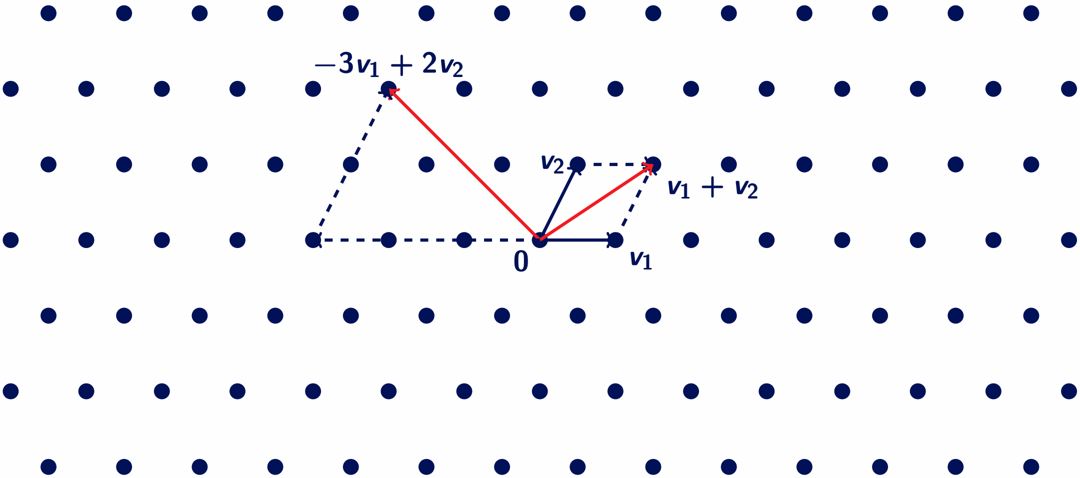
Example $\mathbb{Z} \subset \mathbb{R}$:

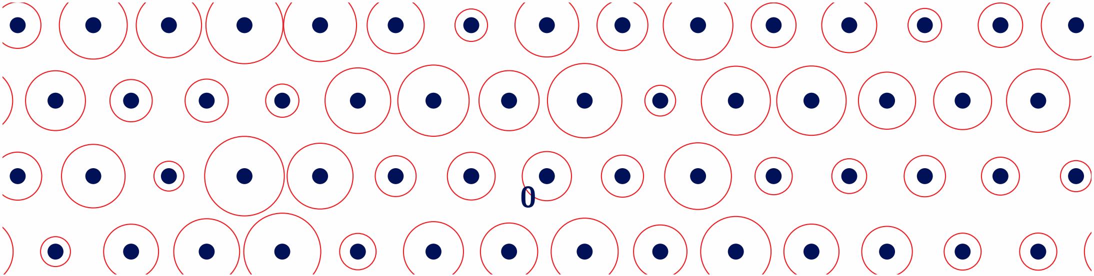



v_2
0
v_1

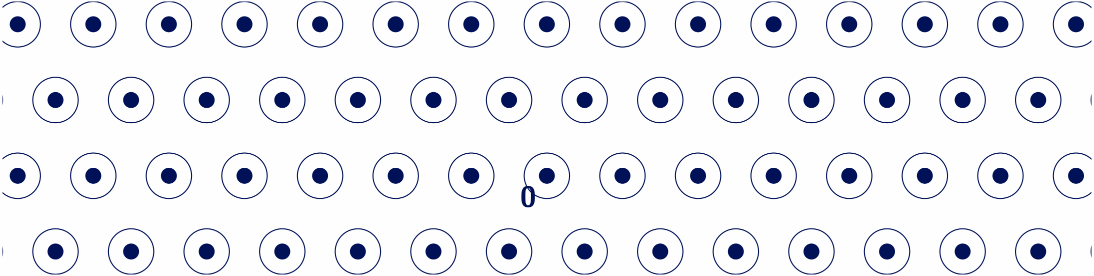
Additive




$-3v_1 + 2v_2$
v_2
$v_1 + v_2$
0
v_1


0

First minimum of a lattice



By the additivity the neighborhood of every lattice point looks the same.

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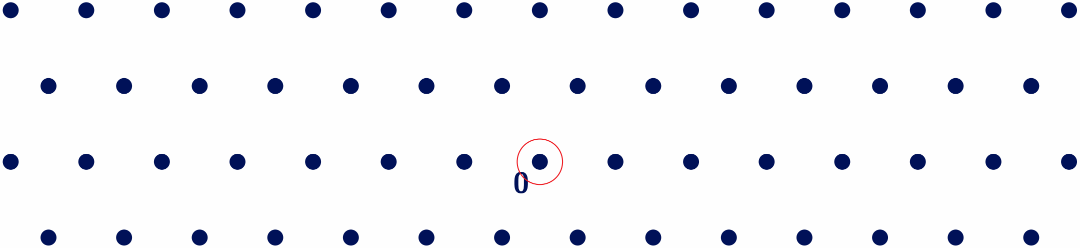

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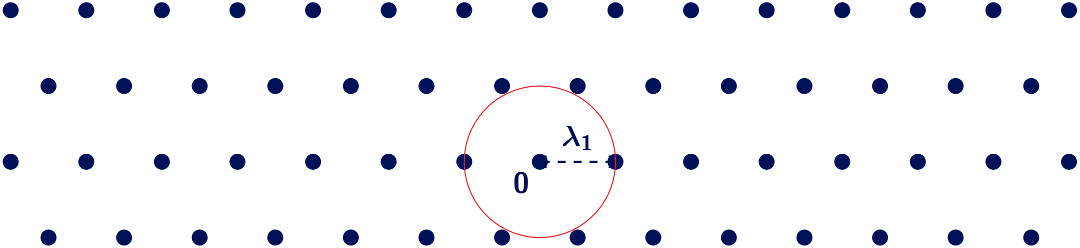
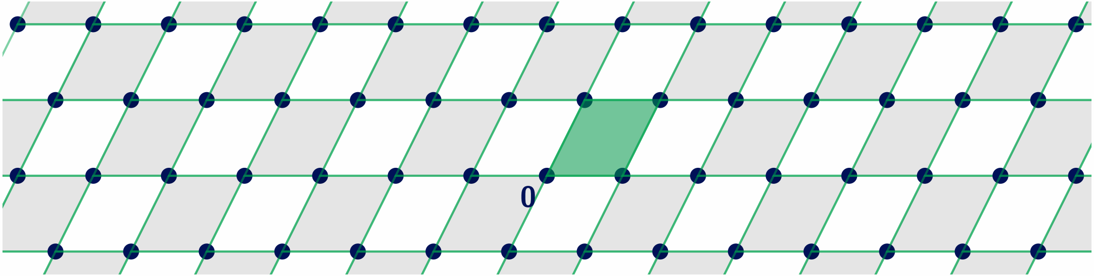


The first minimum $\lambda_1(\mathcal{L})$ of a lattice $\mathcal{L}$ is
the length of the shortest nonzero lattice vector:

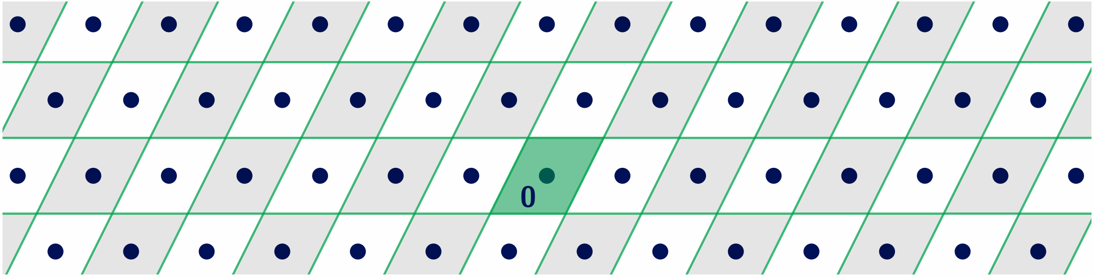
$$\lambda_1(\mathcal{L}) = \min_{x \in \mathcal{L} \setminus \{0\}} \{\|x\|\} > 0.$$

Volume of a lattice



The volume $\mathrm{vol}(\mathcal{L})$ of a lattice $\mathcal{L}$ is the (co-)volume of any fundamental area w.r.t. translation of the lattice:

$$\mathrm{vol}(\mathcal{L}) = \mathrm{vol}(\mathbb{R}^n/\mathcal{L}) \qquad (\mathrm{density}(\mathcal{L}) = 1/\,\mathrm{vol}(\mathcal{L}))$$



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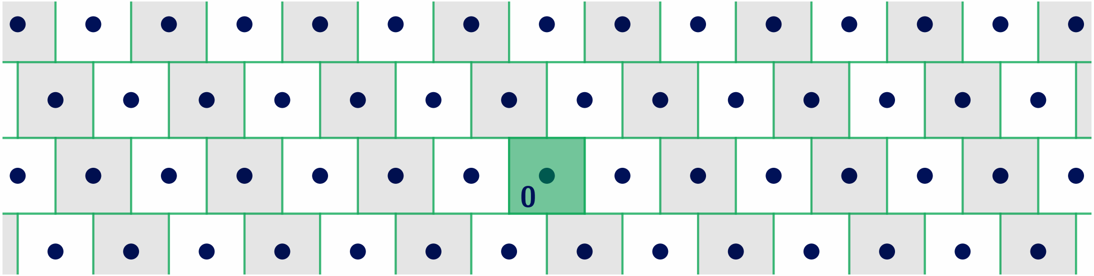

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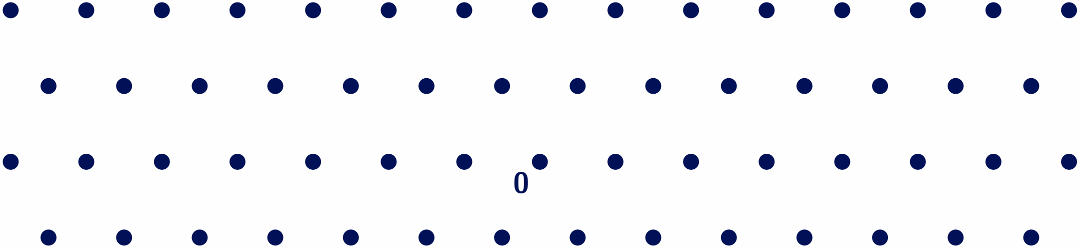

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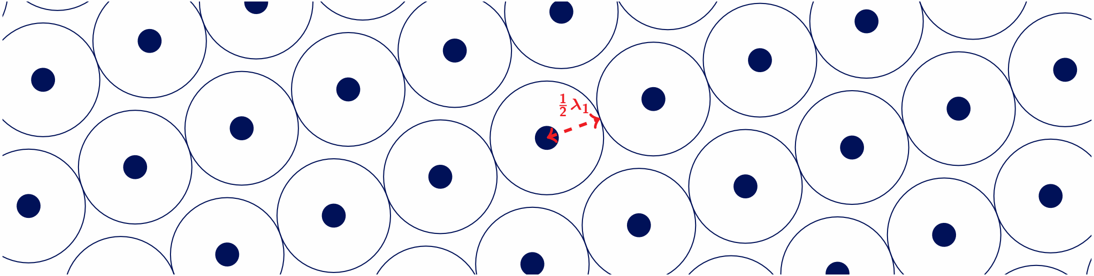
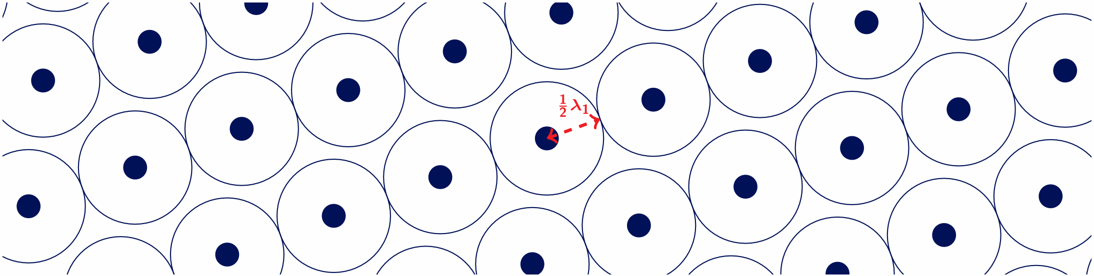


Minkowski's Theorem

For a full-rank lattice $\mathcal{L} \subset \mathbb{R}^n$ we have

$$\mathrm{vol}\left(\frac{1}{2}\lambda_1(\mathcal{L}) \cdot \mathcal{B}^n\right) \leq \mathbf{vol}(\mathcal{L})$$



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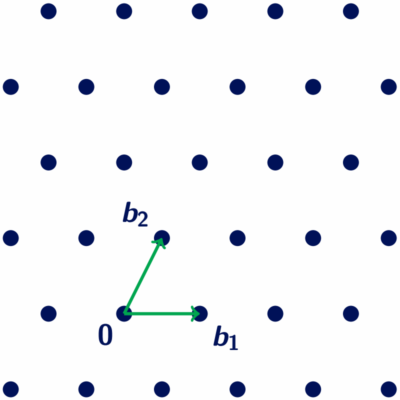
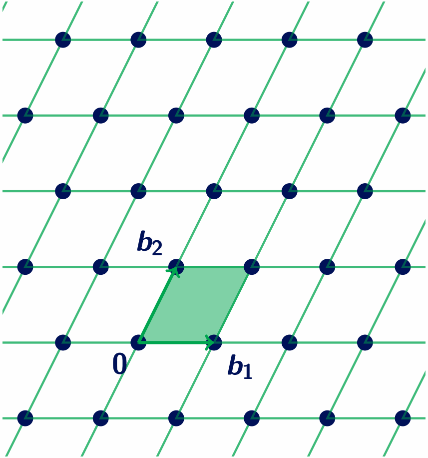
$$\lambda_1(\mathcal{L}) \leq \underbrace{2\frac{\mathrm{vol}(\mathcal{L})^{1/n}}{\mathrm{vol}(\mathcal{B}^n)^{1/n}}}_{\mathrm{Mk}(\mathcal{L})} \approx 2 \cdot \sqrt{n/2\pi e} \cdot \mathrm{vol}(\mathcal{L})^{1/n}$$



Lattice basis

$\mathbb{R}$-linearly independent $\mathbf{b}_1, \ldots, \mathbf{b}_n$

$\mathcal{L}(B) := \{\sum_i x_i \boldsymbol{b}_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n.$



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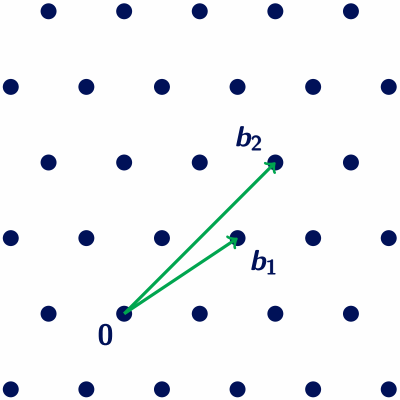
Fundamental Parallelepiped

$\mathcal{P}(B) = B \cdot [0,1)^n$

$\mathbf{vol}(\mathcal{L}) = \mathrm{vol}(\mathcal{P}(B)) = |\det(B)|$



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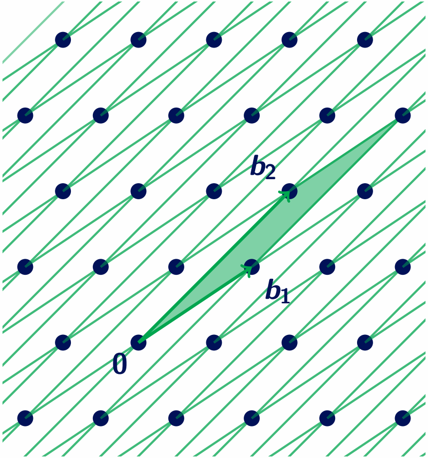
Infinitely many distinct bases

$$B' = B \cdot U \text{ for } U \in \mathcal{GL}_n(\mathbb{Z}).$$



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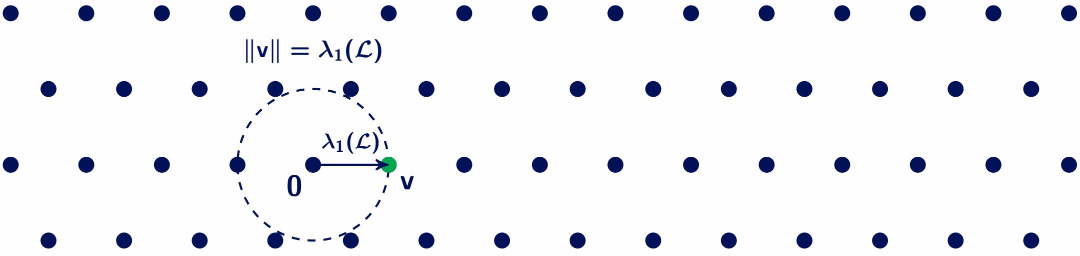

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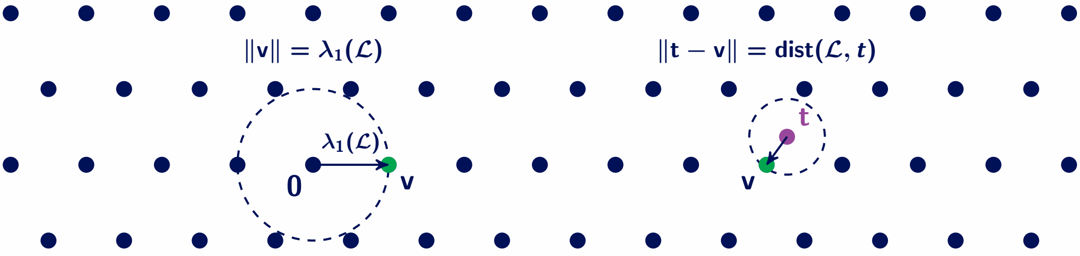
Find a shortest nonzero vector $\boldsymbol{v} \in \mathcal{L}$ of length $\lambda_1(\mathcal{L})$.



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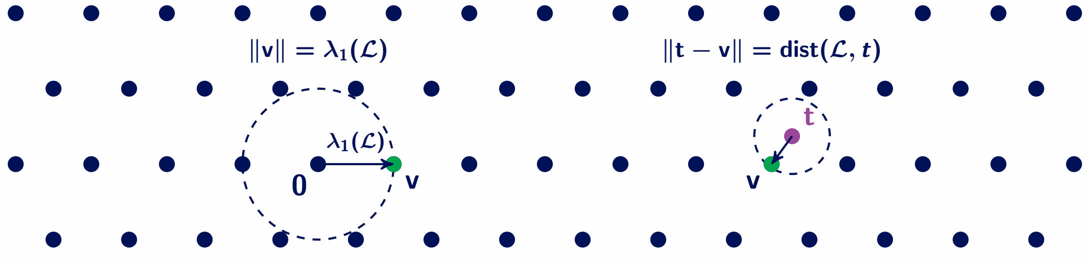

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(even with a quantum computer)

How hard is SVP/CVP?

In theory: best algorithm has asymptotic complexity $2^{c \cdot n + o(n)}$

classical: $c \approx 0.292$, or quantum: $c \approx 0.265$)

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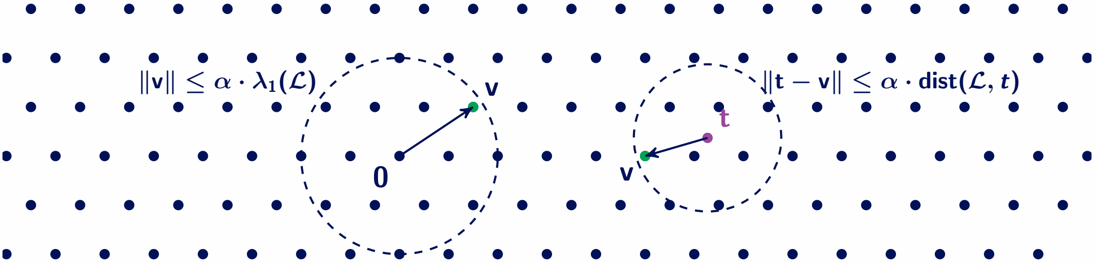
- from $n = 400$ to $n = 1000 \rightsquigarrow$ cryptography

Approximate versions



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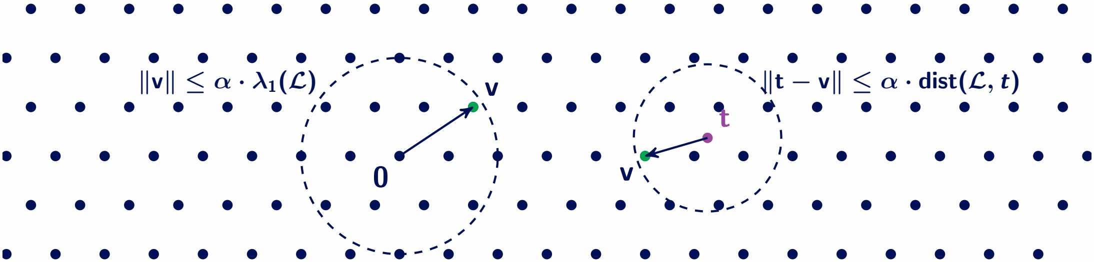

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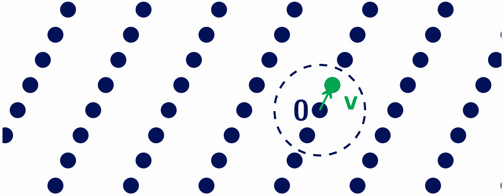
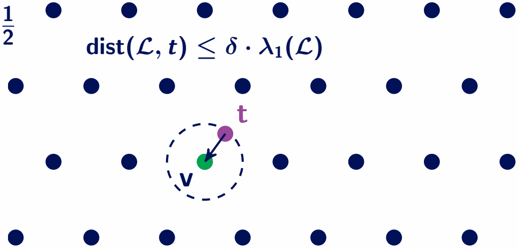
$\lambda_1(\mathcal{L}) \leq \delta \cdot \mathsf{Mk}(\mathcal{L})$



$\delta \leq \frac{1}{2}$

$\mathsf{dist}(\mathcal{L}, t) \leq \delta \cdot \lambda_1(\mathcal{L})$



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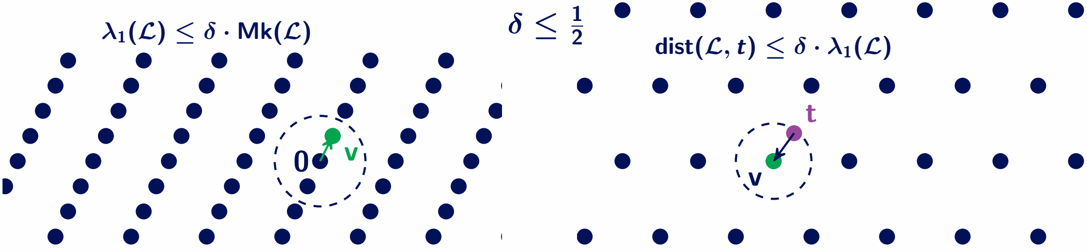
Bounded Distance Decoding (δ-BDD)

CVP with a target unusually close to the lattice.



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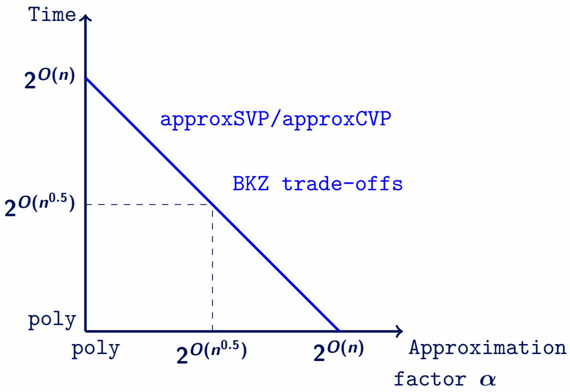

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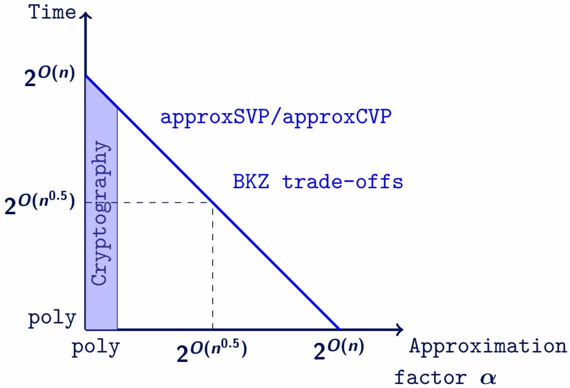

Asymptotic hardness of approx-SVP/CVP

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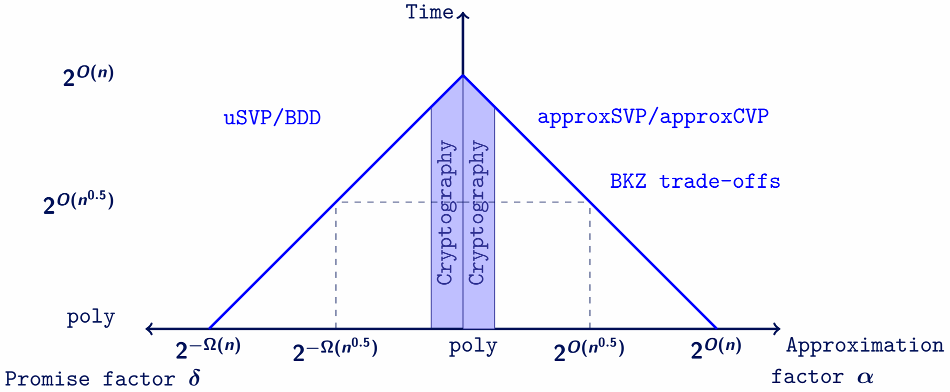

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Many more variants possible: search vs decisional, one vs more solutions, ...)

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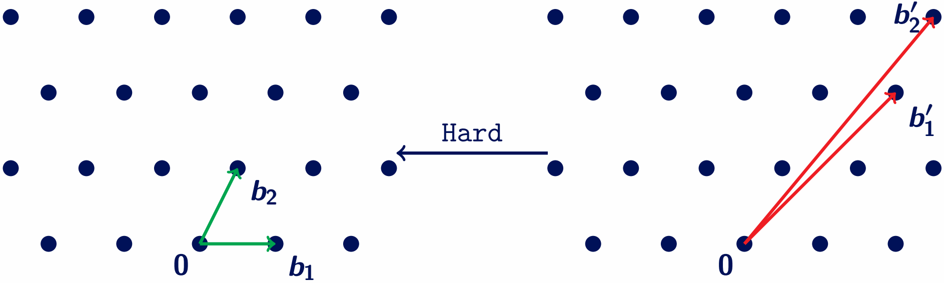
$\boxed{\text{How to build cryptography from this?}}$

Lattice-based cryptography

Good vs bad basis

Good basis (Secret key) | Bad basis (Public key)



Short and close to orthogonal

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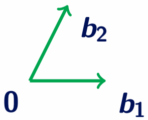

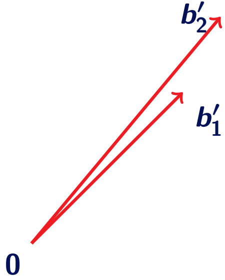

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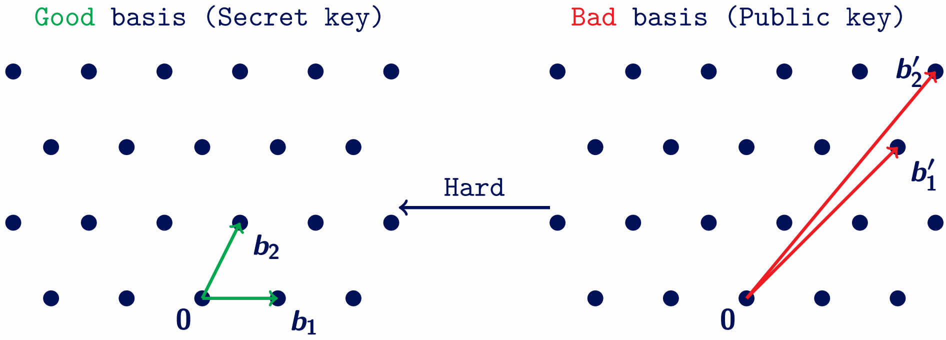
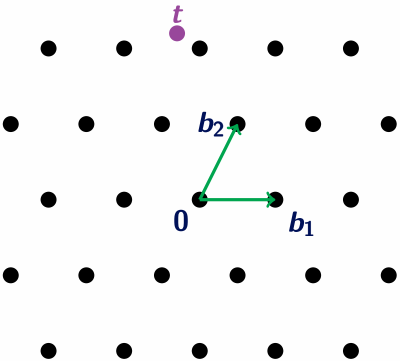


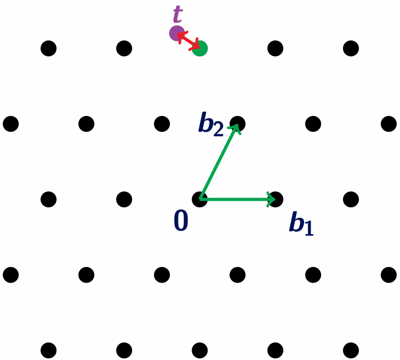
Short and close to orthogonal

Keygen: Generate a random lattice along with a good basis (NTRU, LWE, SIS, ...)



Input: $t = -1.4 \cdot b_1 + 2.2 \cdot b_2$

Solving CVP with a short basis



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$\big\downarrow$ round coordinates

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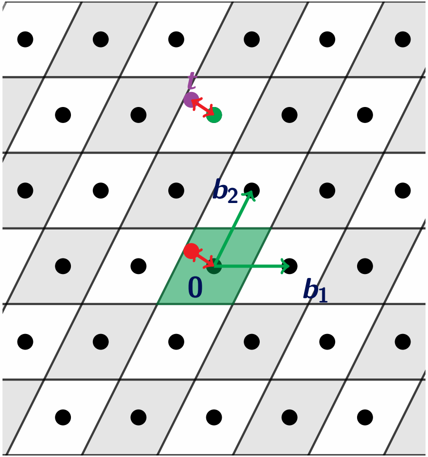

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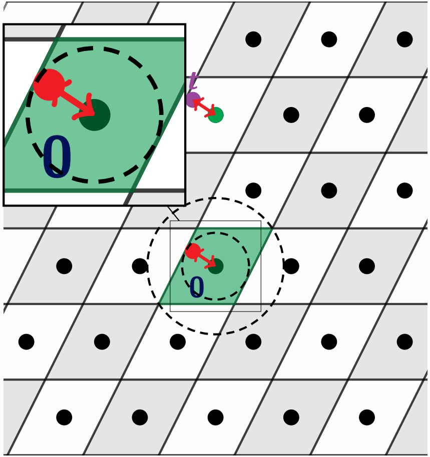
Output: $v = -1 \cdot b_1 + 2 \cdot b_2$

$e = t - v = -.4 \cdot b_1 + 0.2 \cdot b_2$

$e \in B \cdot \left[-\frac{1}{2}, \frac{1}{2}\right)^n$

Solving CVP with a short basis



Input: $t = -1.4 \cdot b_1 + 2.2 \cdot b_2$

$\downarrow$ round coordinates

Output: $v = -1 \cdot b_1 + 2 \cdot b_2$

$$e = t - v = -.4 \cdot b_1 + 0.2 \cdot b_2$$

$$e \in B \cdot \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)^n$$

BDD: inner-radius approxCVP: outer-radius

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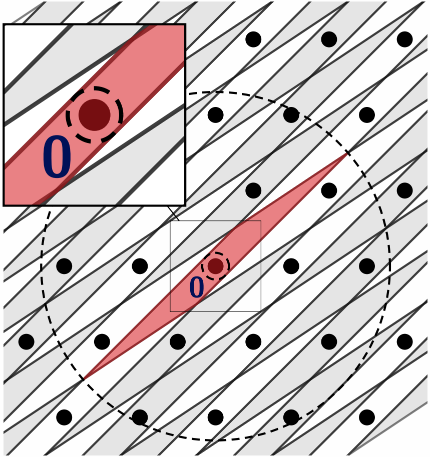

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The better the basis,
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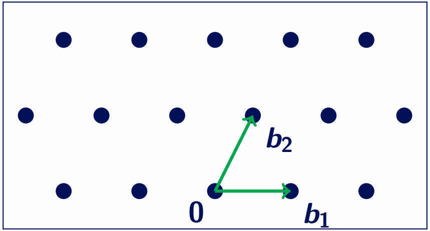

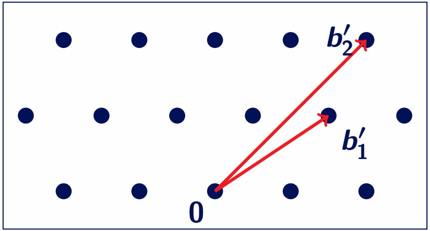

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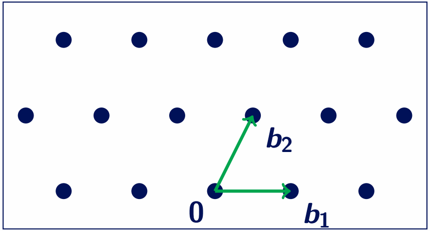

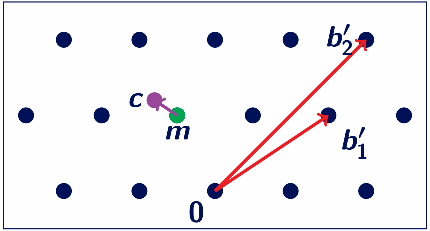

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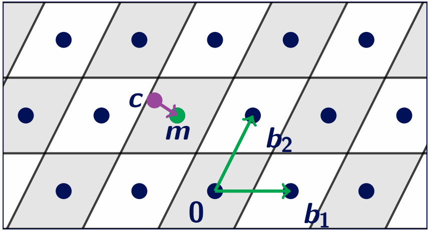

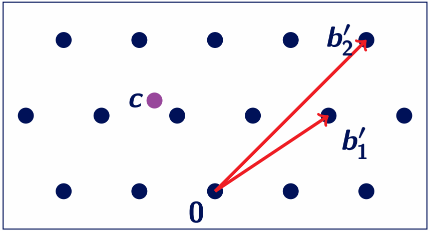

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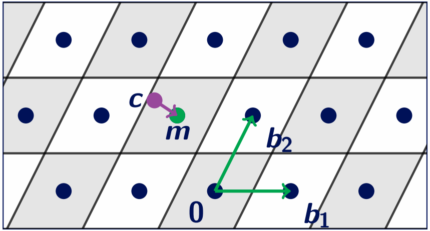
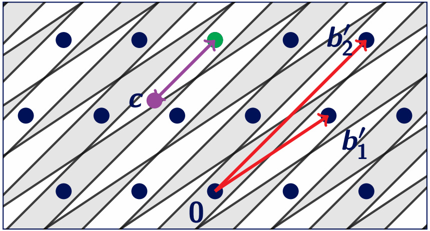
Encryption via BDD





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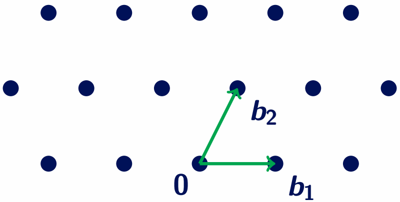

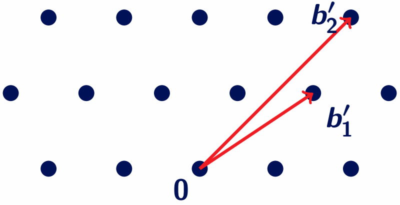

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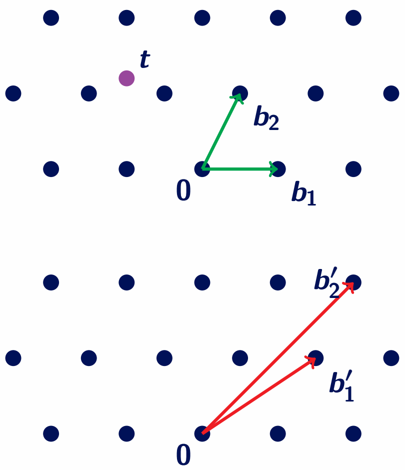

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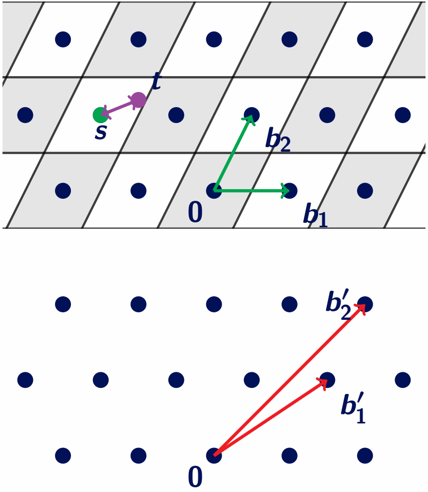

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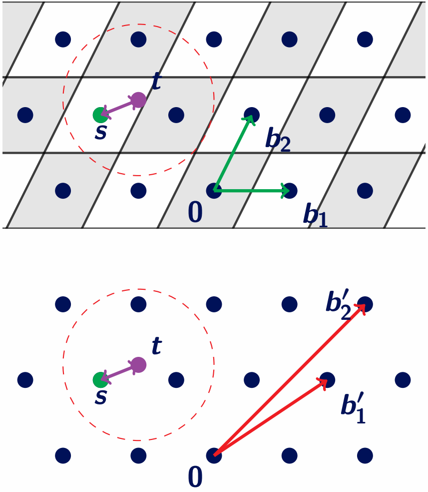

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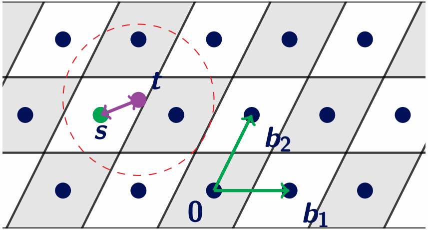
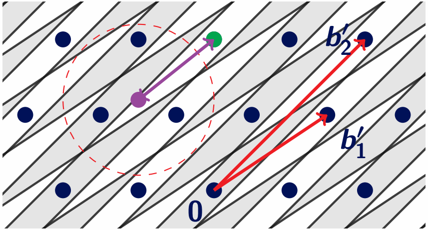
Check that s is close to $H(m)$.

Hash-and-sign signature scheme via approxCVP





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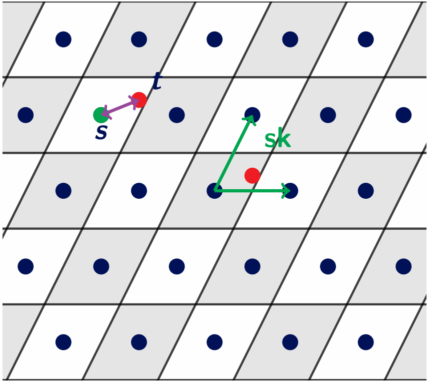
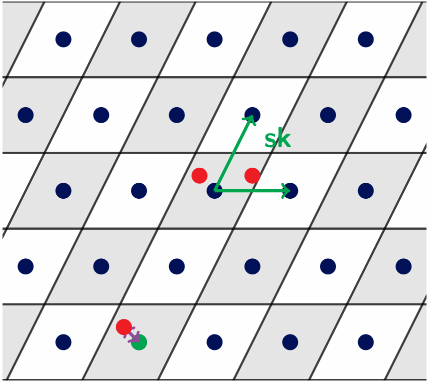
Learning attack on the signature scheme



Parallelepiped attack:

- ask for a signature s on m
- plot $H(m) - s$

[NR06] Nguyen and Regev. Learning a parallelepiped: Cryptanalysis of GGH and NTRU signatures. J. Cryptology



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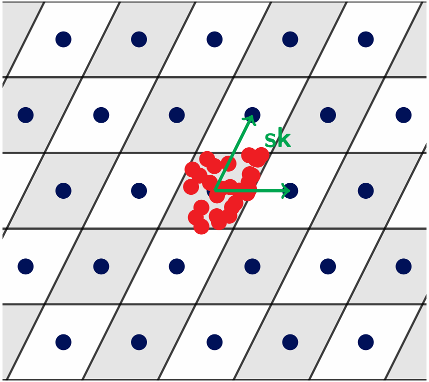

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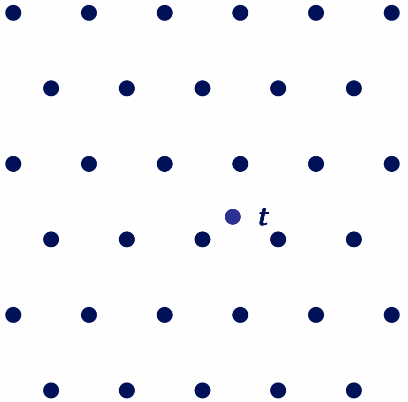
From the shape of the parallelepiped, one can recover the short basis

[NR06] Nguyen and Regev. Learning a parallelepiped: Cryptanalysis of GGH and NTRU signatures. J. Cryptology

Preventing the attack



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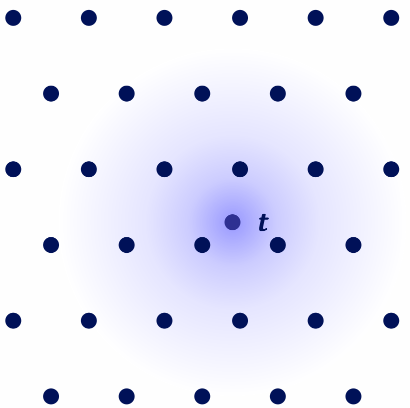

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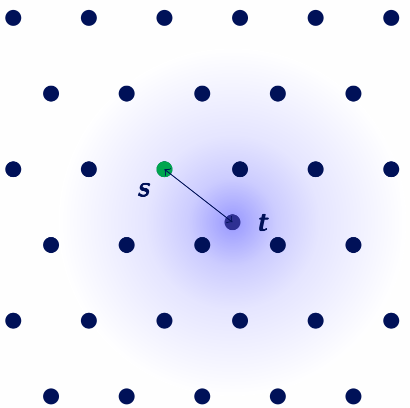

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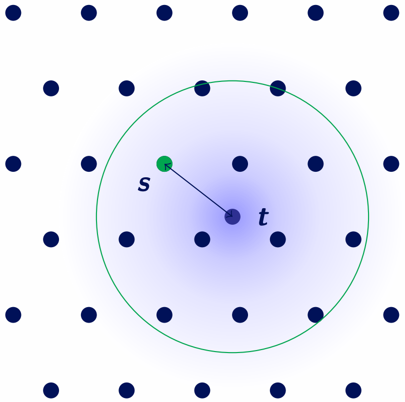

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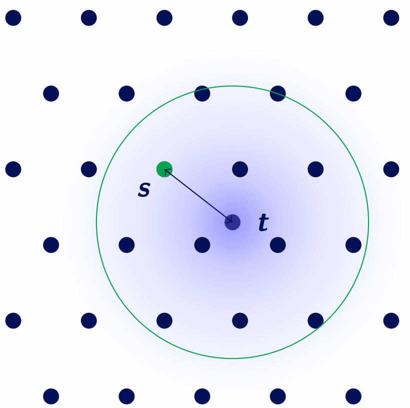

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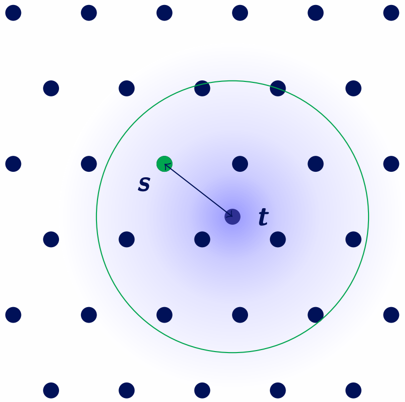

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FALCON = the above + NTRU lattices.

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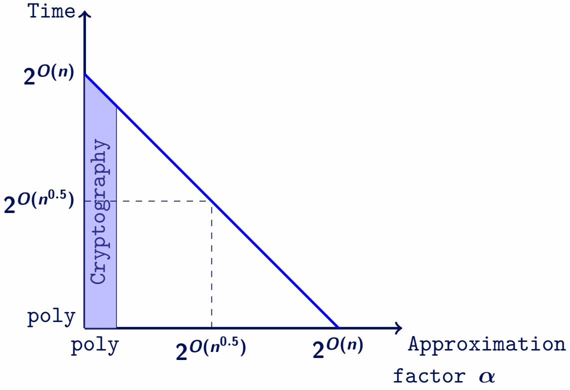
Cryptanalysis - Algorithms to solve (approx)SVP

Best Time/Approximation trade-off for SVP, CVP (even quantumly):



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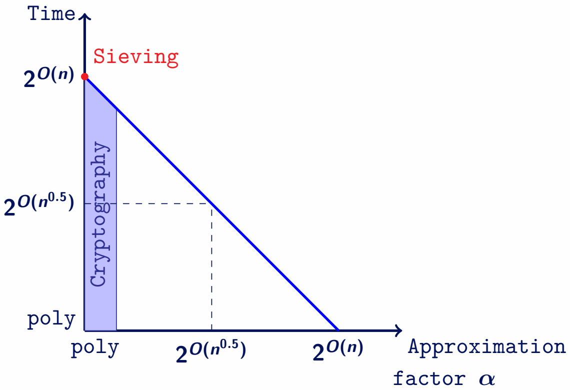

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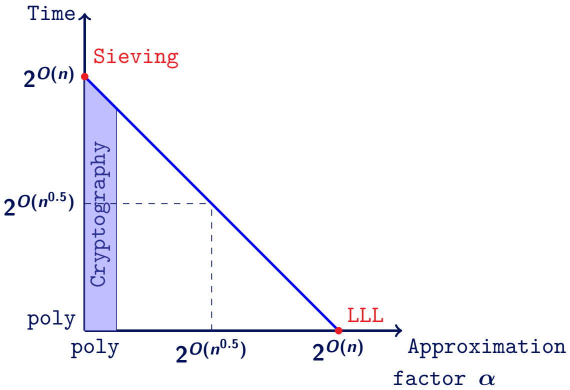

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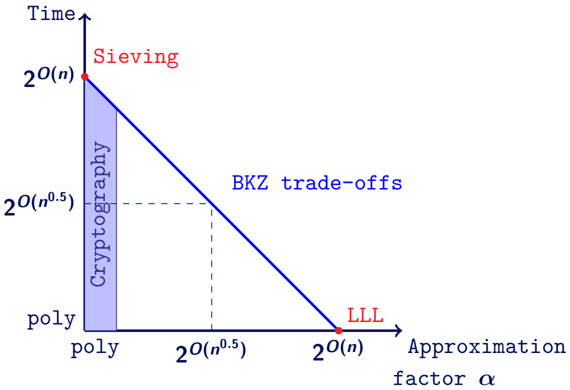

Heuristically solving SVP with lattice sieving

Heuristic assumptions allow to..

Heuristics in lattice-based cryptanalysis

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- bridge the gap between provable and practical algorithms

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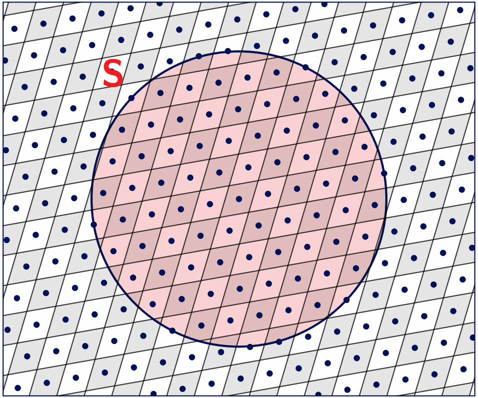

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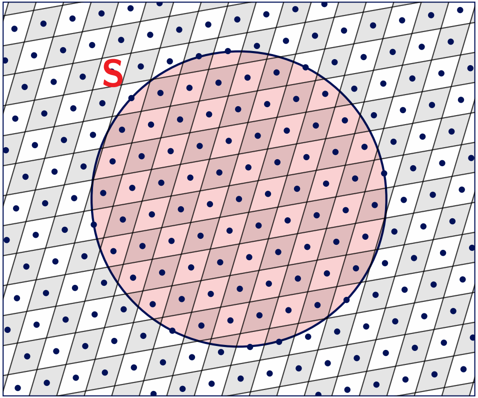

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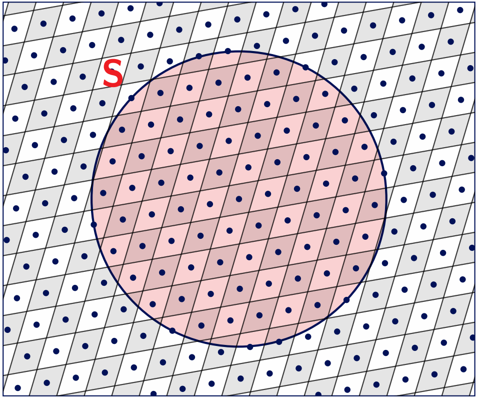

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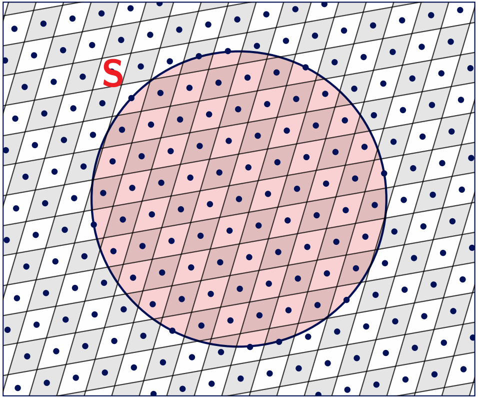

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Intermezzo on high dimensional geometry (1)

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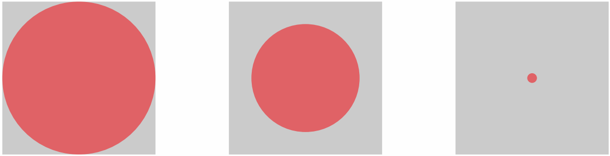

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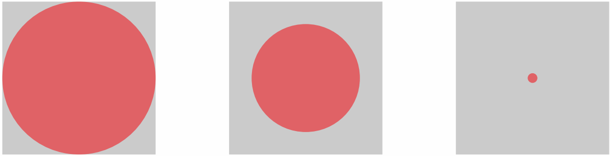

n-dimensional balls with a fixed radius ‘disappear’ for large n.

Intermezzo on high dimensional geometry (2)

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$$\boxed{\lambda_1 \approx \mathrm{gh}(\mathcal{L}) := \frac{\mathrm{vol}(\mathcal{L})^{1/n}}{\mathrm{vol}(\mathcal{B}^n)^{1/n}} \sim \sqrt{n/2\pi e} \cdot \mathrm{vol}(\mathcal{L})^{1/n}.}$$

Scaling by R changes volume by factor R^n.

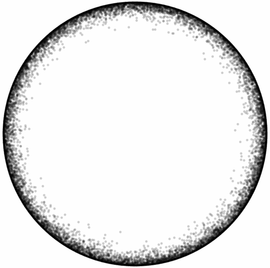
Example: suppose we have a ball $\gamma \cdot \mathcal{B}^{500}$ with the same volume as a 500-dimensional lattice $\mathcal{L} \subset \mathbb{R}^{500}$.

Gaussian Heuristic $\Rightarrow$

$$\left|\left(\gamma \cdot \mathcal{B}^{500} \setminus \{0\}\right) \cap \mathcal{L}\right| \approx 1$$

$$\left|\left(1.05 \cdot \gamma \cdot \mathcal{B}^{500} \setminus \{0\}\right) \cap \mathcal{L}\right| \approx 1.05^{500} = 3.9 \cdot 10^{10}$$

$$\left|\left(0.95 \cdot \gamma \cdot \mathcal{B}^{500} \setminus \{0\}\right) \cap \mathcal{L}\right| = 7.3 \cdot 10^{-12} \approx 0$$



$$\lambda_1(\mathcal{L}) \approx \gamma.$$

$$\lambda_1 \approx \mathrm{gh}(\mathcal{L}) := \frac{\mathrm{vol}(\mathcal{L})^{1/n}}{\mathrm{vol}(\mathcal{B}^n)^{1/n}} \sim \sqrt{n/2\pi e} \cdot \mathrm{vol}(\mathcal{L})^{1/n}.$$

SVP via Lattice Sieving

1. Sample a list $\boldsymbol{L} \subset \mathcal{L}$ of (long) lattice vectors.

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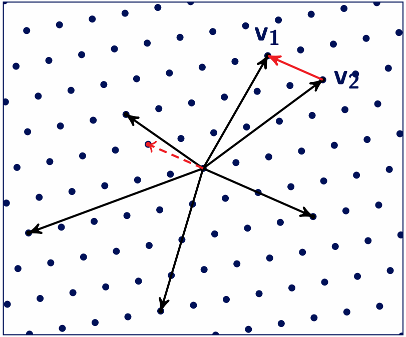
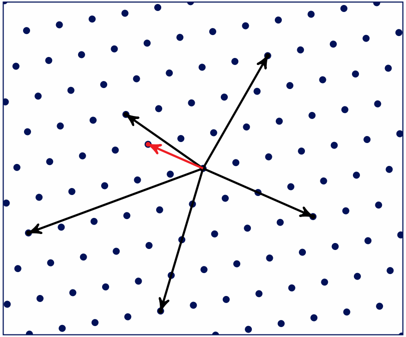
2. Repeat:



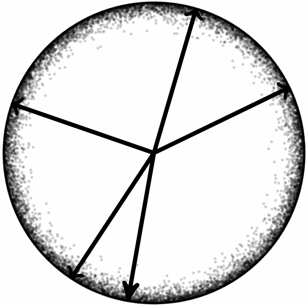
Find close vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{L}$.



Replace $\mathbf{v}_2 \leftarrow \mathbf{v}_1 - \mathbf{v}_2$.

Heuristic complexity analysis

Start with a list $\boldsymbol{L}$ of $\boldsymbol{N}$ vectors of length $\leq \gamma$.



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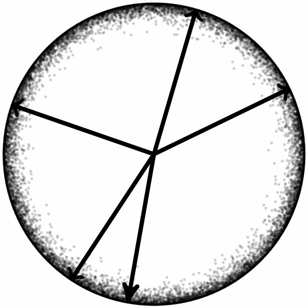

Heuristic assumption

vectors in list L have uniform directions.

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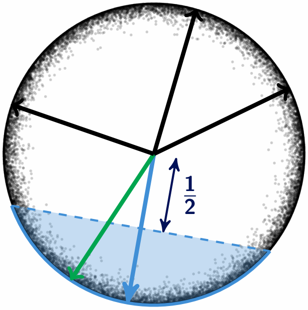
relative volume spherical cap $\approx (3/4 + \epsilon)^{n/2+o(n)}$



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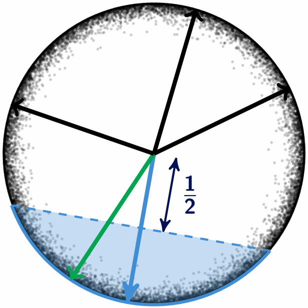

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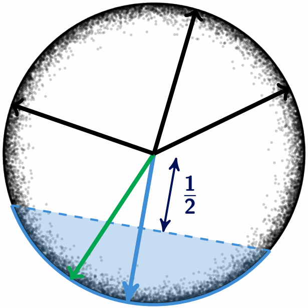

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Space: $N \cdot \mathrm{poly}(n) = (4/3)^{n/2+o(n)} = 2^{0.2075+o(n)}$

Time: $N^2 \cdot \mathrm{poly}(n) = (4/3)^{n+o(n)} = 2^{0.415n+o(n)}$.

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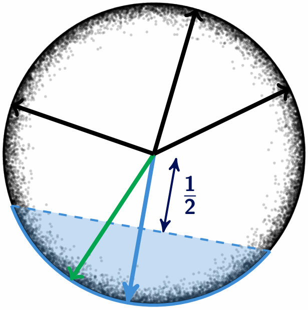

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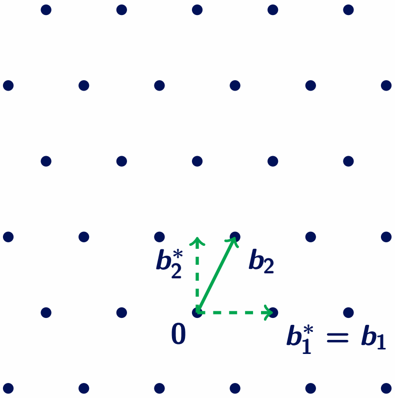
Can be improved to $2^{0.292n+o(n)}$!.

Solving approxSVP/CVP via basis reduction

Gram-Schmidt Orthogonalisation



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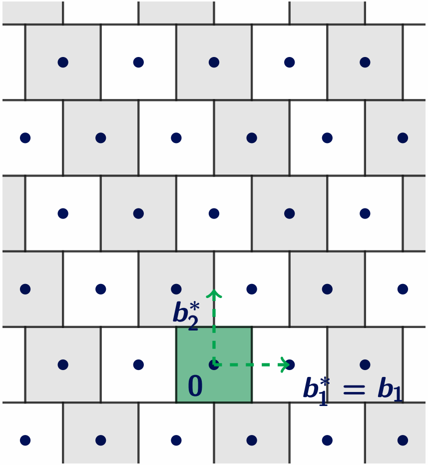

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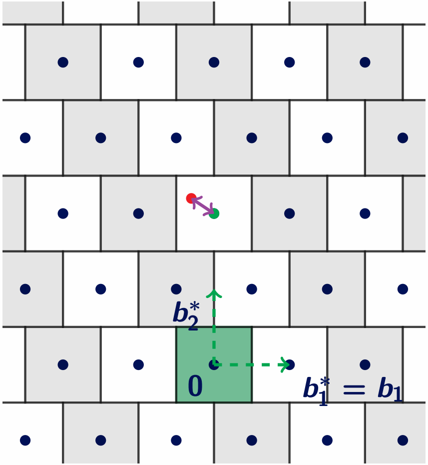

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Nearest plane algorithm

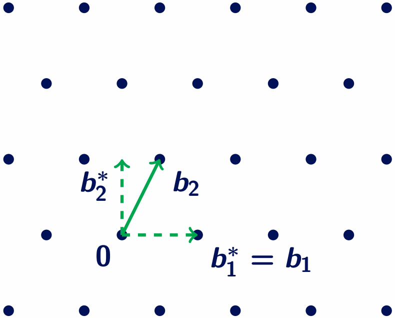
Input: target $t = e$

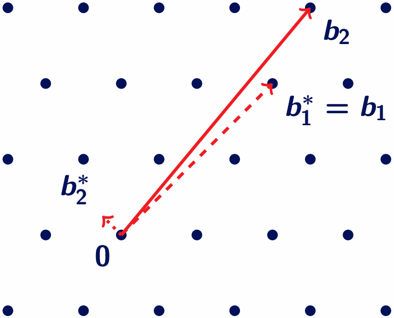
For $j = n, \ldots, 1$:

$$e \leftarrow e - \left\lfloor \frac{\langle e, b_i^* \rangle}{\langle b_i^*, b_i^* \rangle} \right\rceil b_i \, .$$

Output: $e \in \mathcal{F}_{B^*}$

$$b_i^* := \underbrace{\pi_{(b_1,\ldots,b_{i-1})^\perp}}_{\pi_i}(b_i)$$





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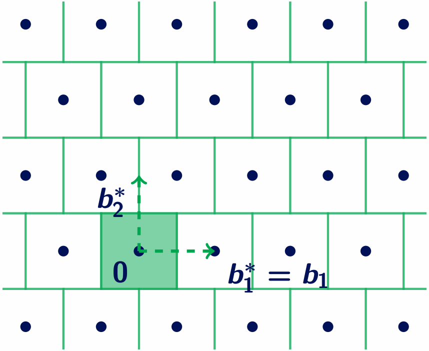

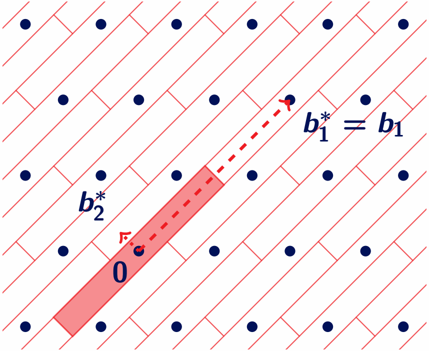

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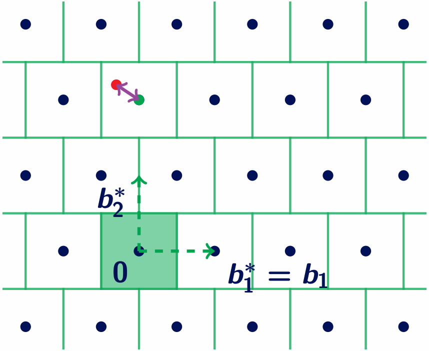

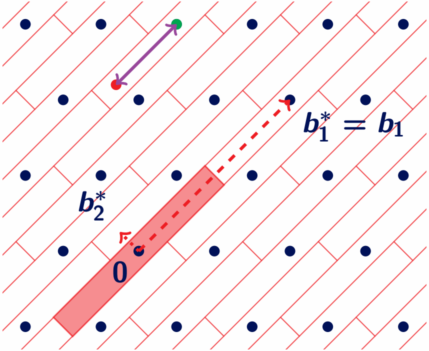

BDD: $\|e\| < \frac{1}{2} \min_i \|b_i^*\|$,

approxCVP: $\|e\|^2 \leq \frac{1}{4} \sum_i \|b_i^*\|^2$.

Good vs Bad basis

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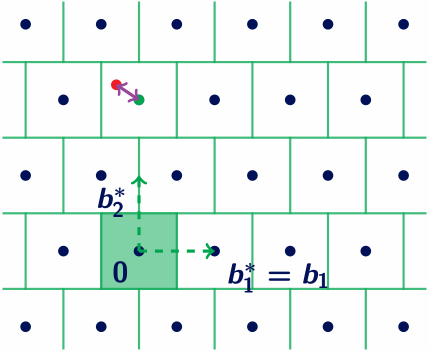

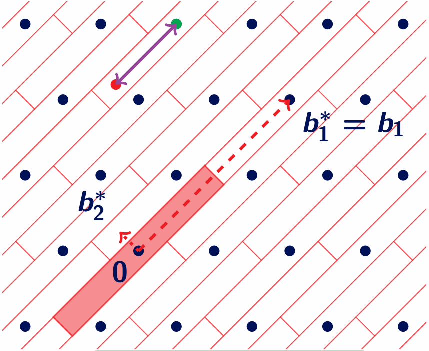

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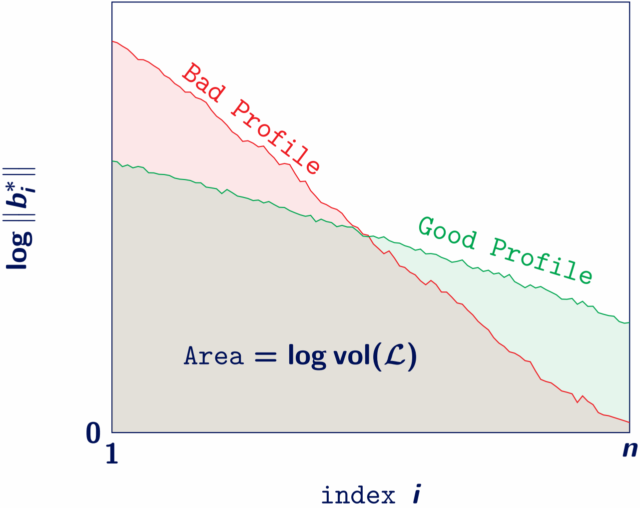
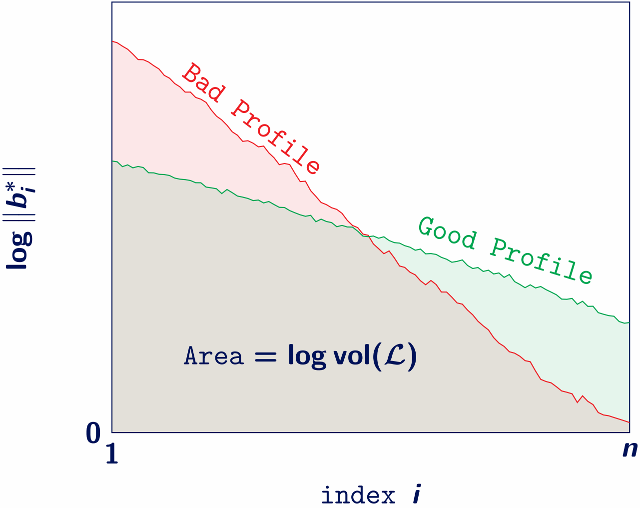
Good Basis:
$\|b_1^*\| \approx \ldots \approx \|b_k^*\|$

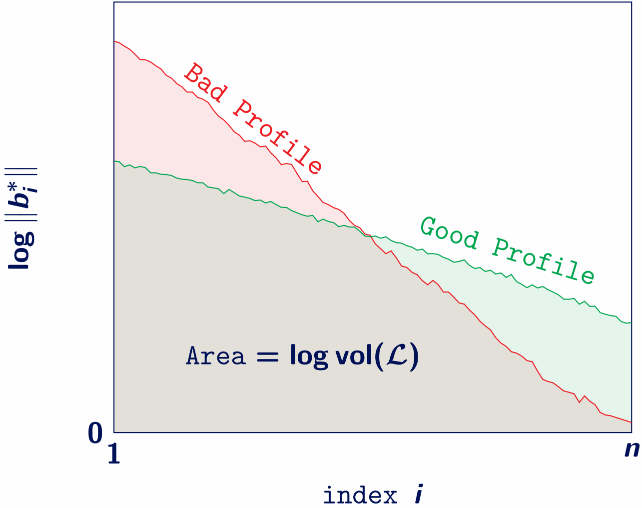
Basis Profile





Basis profile

Measures the length and orthogonality of a basis



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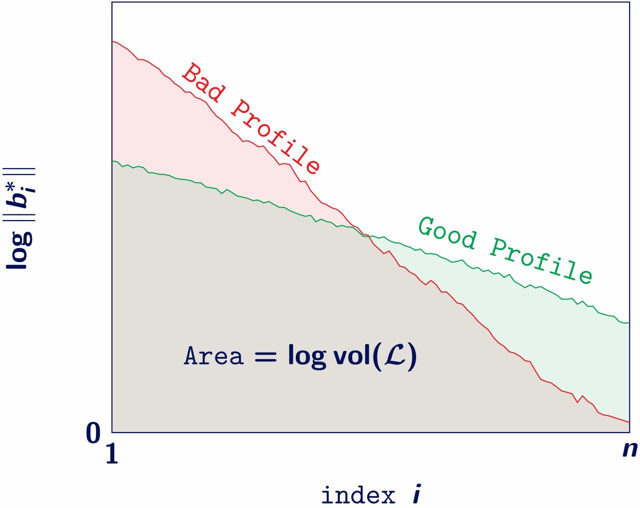
Basis reduction

Input: Bad basis

Output: good basis



Basis profile

Measures the length and orthogonality of a basis

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Input: Bad basis

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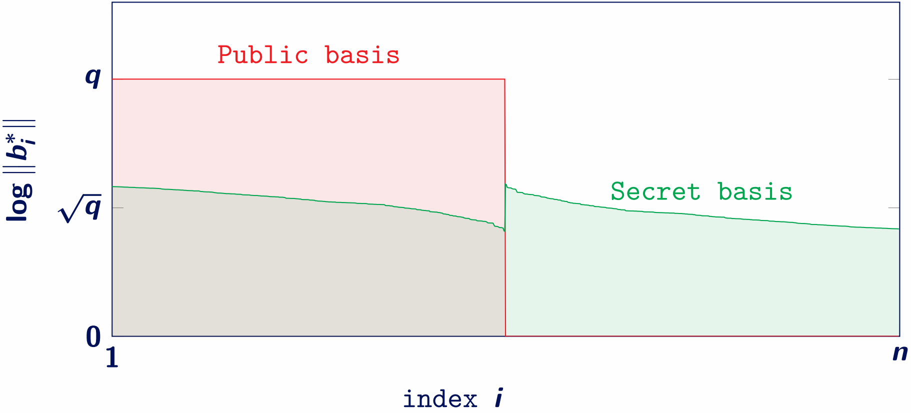
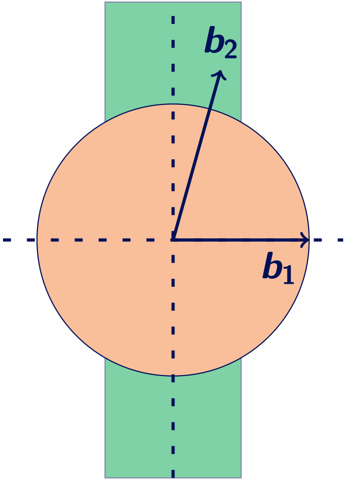
Flatten the profile!

Example: NTRU public vs secret basis

public and secret bases generated from the NTRU problem





Wristwatch Lemma

For any lattice $\mathcal{L}$ of rank 2
there exists a basis (b_1, b_2) s.t.

$$\|b_1\| \leq \|b_2\|$$

$$|\langle b_1, b_2 \rangle| \leq \frac{1}{2}\|b_1\|$$

$$\Downarrow$$

$$\boxed{\|b_1^*\| \leq \sqrt{\frac{4}{3}} \cdot \|b_2^*\|}$$

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A basis B of $\mathcal{L}$ is LLL-reduced if $(\pi_i(b_i), \pi_i(b_{i+1}))$ is Lagrange Reduced for all $i < n$.

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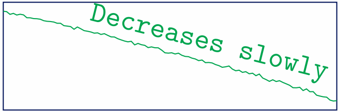
$\log \left\| b_i^* \right\|$



index i

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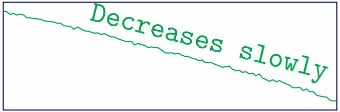

index i

LLL Reduction

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Algorithm

While $\exists i$ s.t. $(\pi_i(b_i), \pi_i(b_{i+1}))$ is not Lagrange Reduced, Langrange Reduce it.

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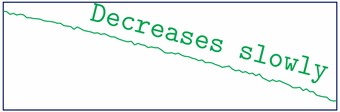

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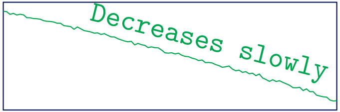

index i

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Termination in poly-time:

Requires a slight relaxation. (ϵ-Lagrange Reduced)

Proof argument:

$$P = \sum_{i \leq n} (n+1-i) \cdot \log \left\|b_i^*\right\|$$

Decreases by ϵ at each step and is lower-bounded.

BKZ algorithm

- Define the projected sublattice basis $\boldsymbol{B}_{l:r} := (\pi_l(\boldsymbol{b}_l), \ldots, \pi_l(\boldsymbol{b}_{r-1}))$.

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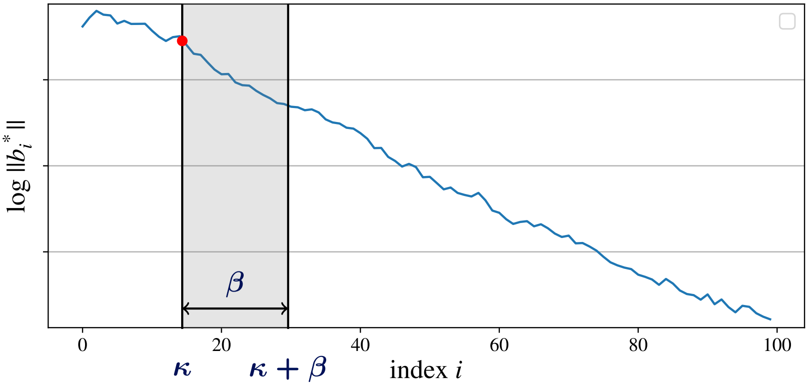

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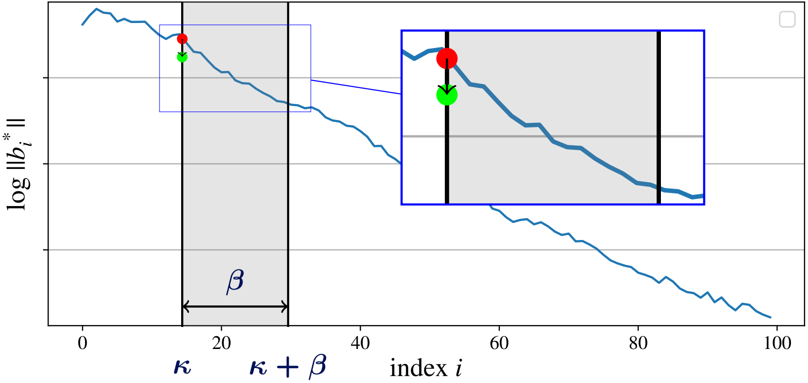

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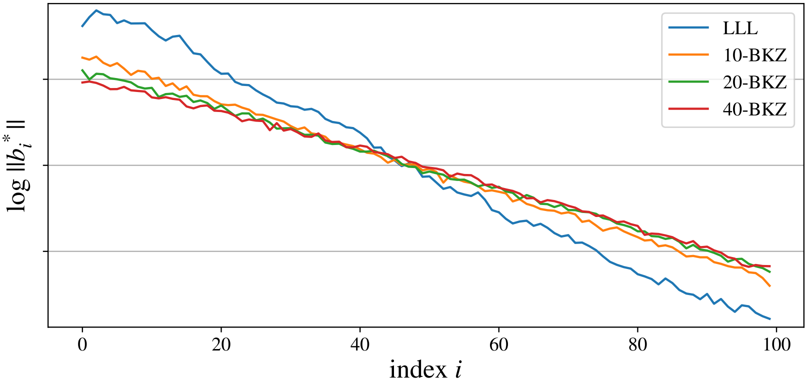
- Reduction better for larger blocksize β, but cost $2^{0.292\beta+o(n)}$.



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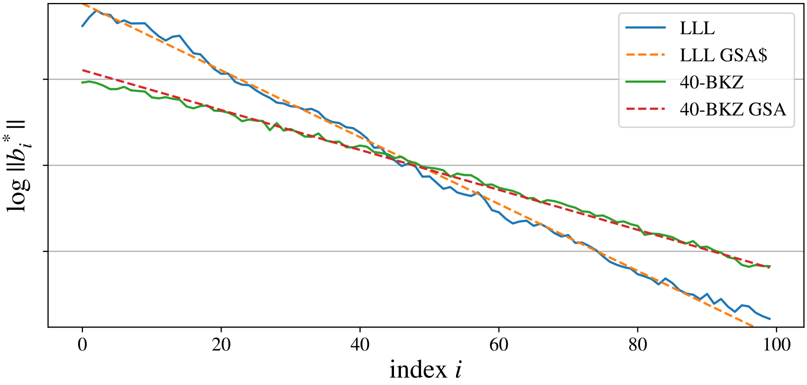
- Reduction better for larger blocksize β, but cost $2^{0.292\beta+o(n)}$.
- Behaviour well understood for 'random' lattices. [GSA]



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$$\boxed{\begin{array}{c}\text{SVP for rank } 2 \\ \text{(Lagrange reduction)}\end{array}} \longrightarrow \boxed{\|b_1\| \leq \sqrt{4/3}^{\frac{n-1}{2}} \cdot \mathrm{vol}(\mathcal{L})^{1/n}}$$

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SVP for rank β (sieving)	$\longrightarrow$	$\|b_1\| \leq O(\beta)^{\frac{n-1}{2(\beta-1)}} \cdot \mathrm{vol}(\mathcal{L})^{1/n}$

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SVP for rank β
(sieving)

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- Same algorithms also solve promise variants uSVP and BDD

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- More efficiency: algebraic lattices (ideal and module lattices)

Part II

Plan

Part I

Lattice theory

- Lattices
- Hard problems

Cryptography

- Trapdoor bases
- Encryption, Signature

Cryptanalysis

- Lattice Sieving
- Basis Reduction

Part II

Lattices used in cryptography

- SIS, LWE, decLWE
- Security proofs

Hardness Reductions

- search to decision
- WC to AC reductions

Algebraic Lattices

- Ideal and module lattices
- NTRU, RLWE, mod-LWE

Limitations of SVP (and CVP)

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- no efficient algorithm that works for any lattice
- but for some lattice it might be easier

Limitations of SVP (and CVP)

SVP and CVP are hard in the worst case

- no efficient algorithm that works for any lattice
- but for some lattice it might be easier

For crypto, we need problems that are hard on average

(i.e., for a random instance, the problem is hard with overwhelming probability)

random q-ary lattices

q-ary lattices

Notations: q, n, m integers, $1 \leq n \ll m$, $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$

- A lattice $\mathcal{L} \subset \mathbb{R}^m$ of dimension m is called q-ary if

$$q\mathbb{Z}^m \subset \mathcal{L} \subset \mathbb{Z}^m.$$

- Let $A \in \mathbb{Z}_q^{m \times n}$, then we define the row-generated q-ary lattice

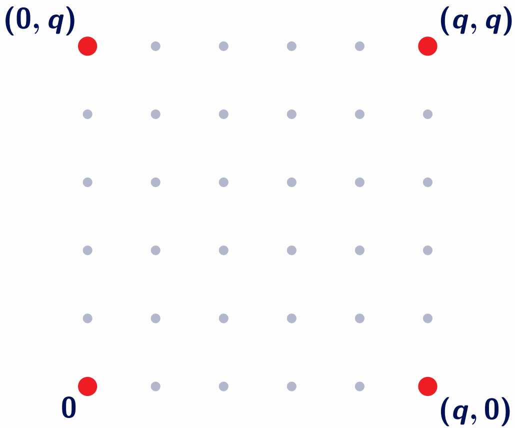
$$\Lambda_q(A) := \{y \in \mathbb{Z}^m : y \equiv Ax \bmod q \text{ for some } x \in \mathbb{Z}_q^n\} = A\mathbb{Z}^n + q\mathbb{Z}^m$$

- and the parity-check q-ary lattice

$$\Lambda_q^{\perp}(A) := \{x \in \mathbb{Z}^m : x^\top A \equiv 0 \bmod q\} = \ker(A^\top : \mathbb{Z}^m \to \mathbb{Z}_q^n)$$

- Exercise: if q prime and A has full column-rank, then

$$\mathrm{vol}(\Lambda_q(A)) = q^{m-n}, \quad \mathrm{vol}(\Lambda_q^{\perp}(A)) = q^n$$

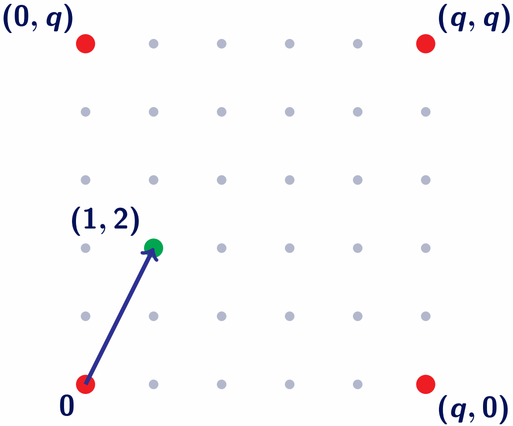


Suppose $q = 5, n = 1, m = 2$,

$$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Lambda_q(A) = A\mathbb{Z}^n + q\mathbb{Z}^m = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \mathbb{Z} + 5\mathbb{Z}^2$$

Example



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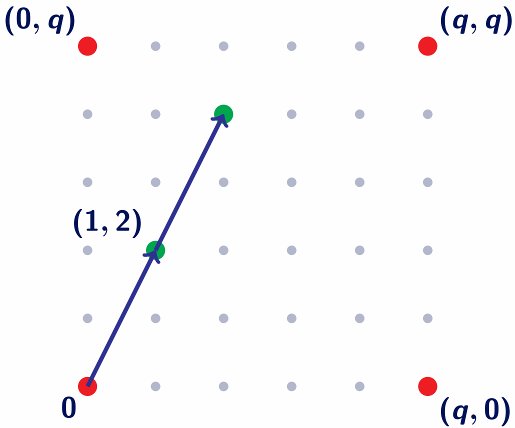

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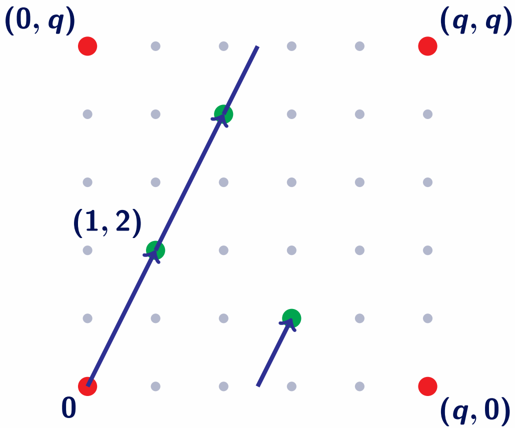

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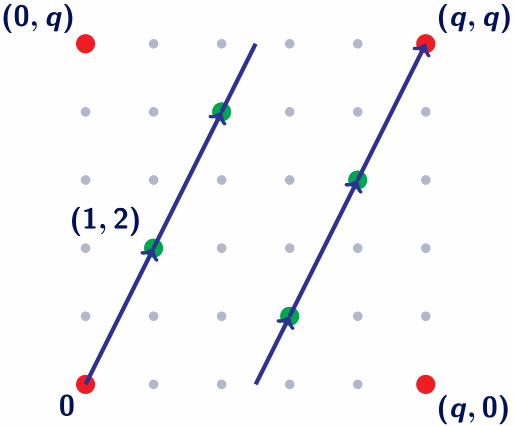

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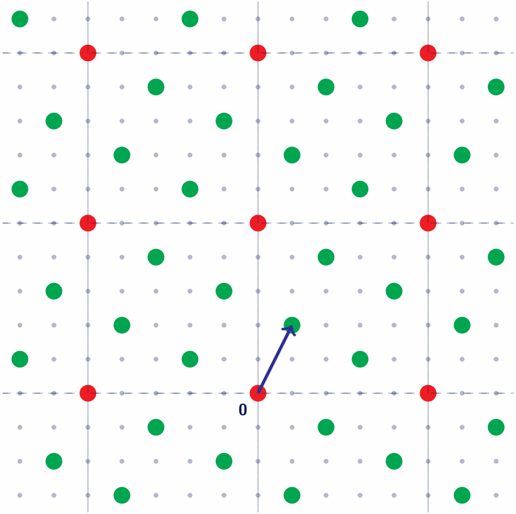

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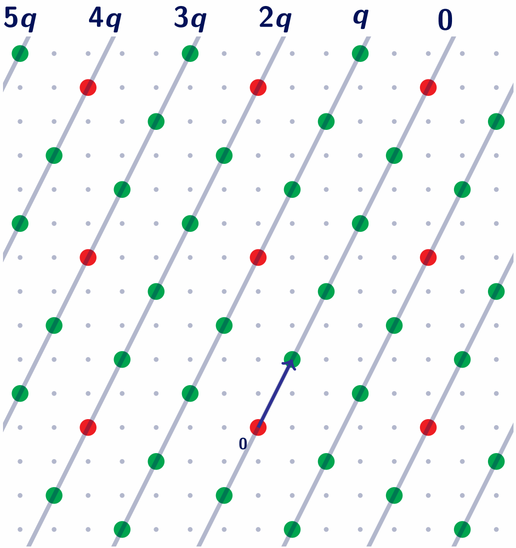

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Parity-check representation:

$$\Lambda_q\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \Lambda_q^{\perp}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right)$$

$$= \{(x, y) \in \mathbb{Z}^2 : -2x + y \equiv 0 \bmod q\}$$

Family of random q-ary lattices

- Random $\boldsymbol{q}$-ary lattice: sample $\boldsymbol{A} \in \mathcal{U}\left(\mathbb{Z}_{\boldsymbol{q}}^{\boldsymbol{m}\times\boldsymbol{n}}\right)$, and consider $\boldsymbol{\Lambda}_{\boldsymbol{q}}(\boldsymbol{A})$

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- Defines average-case problems!

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- Defines average-case problems!
- For $X \in \{$approxSVP, approxCVP, uSVP, BDD$\}$ and $m = \mathrm{poly}(n)$ we have

Solving X in any lattice of rank m	$\geq$	Solving X with non-negligible prob. in a random q-ary lattice

- These average-case problems are also known as (I)SIS and LWE.

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Worst-case to average-case reduction

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Worst-case to average-case reduction

- These average-case problems are also known as (I)SIS and LWE.

Notations: q, B integers, $1 \leq B \ll q$, $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$

SIS (Short Integer Solution) [Ajt96]

Parameters: B and q

Problem: Given $\boxed{A} \leftarrow \mathcal{U}(\mathbb{Z}_q^{m\times n})$ (with $n \log q < m$)

Find $x \in \mathbb{Z}^m$ s.t. $\boxed{x^T}\,\boxed{A} = 0 \bmod q$ with $\|x\| \leq B$ and $x \neq 0$.

[Ajt96] Ajtai. Generating hard instances of lattice problems. STOC.

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Solving SIS with non-negligible probability	$\gtrsim$	Solving approx-SVP in any lattice of rank n

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The SIS problem

Notations: q, B integers, $1 \leq B \ll q$, $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$

ISIS (Inhomogeneous Short Integer Solution) [Ajt96]

Parameters: B and q

Problem: Given $\boxed{A} \leftarrow \mathcal{U}(\mathbb{Z}_q^{m\times n})$, $\boxed{y} \leftarrow \mathcal{U}(\mathbb{Z}_q^n)$

Find $x \in \mathbb{Z}^m$ s.t. $\boxed{x^T}\,\boxed{A} = \boxed{y^T} \bmod q$ with $\|x\| \leq B$.

Solving approx-CVP in any lattice lattice of rank m	$\geq$	Solving ISIS with non-negligible probability	$\gtrsim$	Solving approx-SVP in any lattice of rank n

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(I)SIS is as hard as worst-case lattice problems

Theorem [Ajt96]

For any $m = \text{poly}(n)$ and $B > 0$ and sufficiently large $q \geq B \cdot \text{poly}(n)$, it holds that solving SIS is at least as hard as solving γ-SIVP on arbitrary n-dimensional lattice, for some approximation factor $\gamma = B \cdot \text{poly}(n)$.

(SIVP = shortest independent vectors problems.

Objective: find n short linearly independent vectors in the lattice)

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- the **poly** quantities have been improved in more recent works
- for typical parameters: SIS $\cong$ ISIS
- see [Pei16] for a survey

[Pei16] Peikert. A decade of lattice cryptography. Foundations and trends in theoretical computer science

SIS is a lattice problem

SIS (Short Integer Solution)

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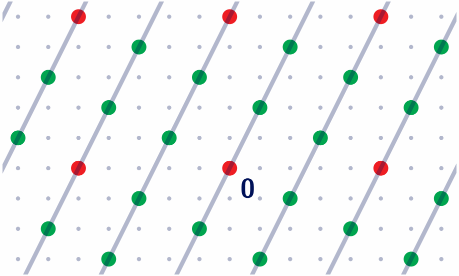

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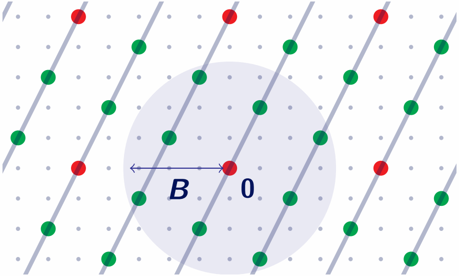

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SIS $\approx$ approx-SVP in random $\Lambda_q^{\perp}(A)$

Average-case approx-SVP problem

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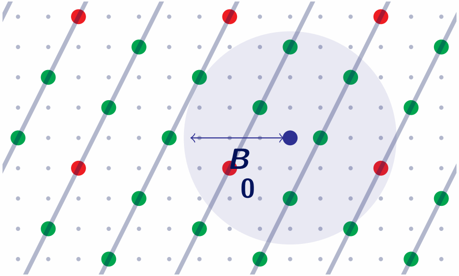

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Trapdoor basis

Lemma [Ajt99]

One can efficiently create a uniform SIS lattice $\boldsymbol{\Lambda_q^\perp(A)}$ together with a short basis of it.

[Ajt99] Ajtai. Generating hard instances of the short basis problem. ICALP.

Trapdoor basis

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Idea: start with a short basis, then perturb and randomize it

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Trapdoor basis

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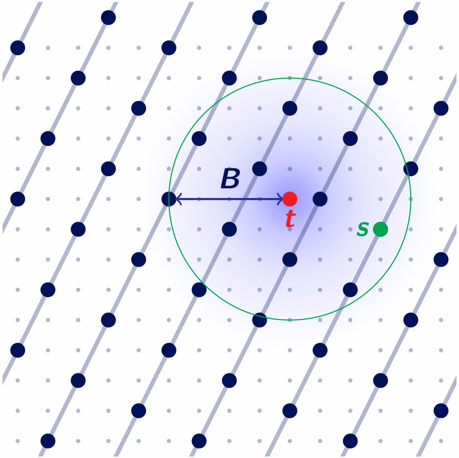
$$\begin{array}{ccc} & \nearrow & \textbf{pk} = \text{uniform random } A \\ \text{SIS } \textbf{KeyGen} & & \Big\downarrow \geq \text{ SIS problem} \\ & \searrow & \textbf{sk} = \text{good basis } S \text{ of } \Lambda_q^{\perp}(A) \end{array}$$

[Ajt99] Ajtai. Generating hard instances of the short basis problem. ICALP.

Hash-and-sign signature scheme from SIS

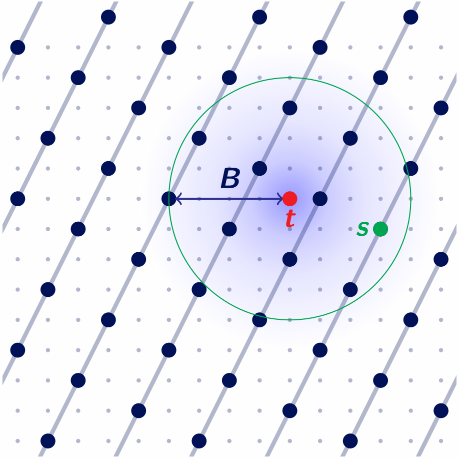


Sign: hash message to $\boldsymbol{t} \in \mathbb{Z}_q^m$,

sample nearby $\boldsymbol{s} \in \Lambda_q^{\perp}(\boldsymbol{A})$ with sk

Verify: $\boldsymbol{s} \in \Lambda_q^{\perp}(\boldsymbol{A}) \wedge \|\boldsymbol{t} - \boldsymbol{s}\| \leq B$

Hash-and-sign signature scheme from SIS



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Security proof

key-recovery $\geq$ SIS problem

signature forgery $\geq$ ISIS problem

(assuming no leakage from sampling

can be proven in Random Oracle Model) .

Signature scheme based on hard average-case lattice problem

The LWE problem

LWE (Learning With Errors) [Reg05]

Sample $\boxed{\boldsymbol{A}} \leftarrow \mathcal{U}(\mathbb{Z}_q^{m\times n})$, $\boxed{\boldsymbol{s}} \leftarrow \mathcal{U}(\mathbb{Z}_q^n)$ and $\boxed{\boldsymbol{e}} \leftarrow \mathcal{U}(\{-B, \cdots, B\}^m)$

Given $\boxed{\boldsymbol{A}}$ and $\boxed{\boldsymbol{b}}$, where $\boxed{\boldsymbol{b}} := \boxed{\boldsymbol{A}}\,\boxed{\boldsymbol{s}} + \boxed{\boldsymbol{e}} \bmod q$

Recover $\boldsymbol{s}$ or $\boldsymbol{e}$

[Reg05] Regev. On lattices, learning with errors, random linear codes, and cryptography. STOC.

The LWE problem

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Recover s or e

Remark. Sometimes $\boxed{s}$ is small in $\mathbb{Z}_q$ (not uniform)

- this is (almost) equivalent
- prove it (*hint:* you are allowed to change m)

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Recover s or e

Solving LWE with non-negligible probability	$\gtrsim$ quantumly!	Solving approx-SVP in any lattice of rank n

[Reg05] Regev. On lattices, learning with errors, random linear codes, and cryptography. STOC.

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Given $\boxed{A}$ and $\boxed{b}$, where $\boxed{b} := \boxed{A}\,\boxed{s} + \boxed{e} \bmod q$

Recover s or e

Solving BDD in any lattice of rank m	$\gtrsim$	Solving LWE with non-negligible probability	$\gtrsim$ quantumly!	Solving approx-SVP in any lattice of rank n

[Reg05] Regev. On lattices, learning with errors, random linear codes, and cryptography. STOC.

LWE is quantumly as hard as worst-case lattice problems

Theorem [Reg05]

For any $m = \mathrm{poly}(n)$, modulus $q \leq 2^{\mathrm{poly}(n)}$ and $B \geq 2\sqrt{n}$, solving LWE is at least as hard as quantumly solving γ-SIVP on arbitrary n-dimensional lattice, for some approximation factor $\gamma = \tilde{O}(n \cdot q/B)$.

❦ the reduction is for a variant of LWE where s and e are sampled from a discrete Gaussian distribution of parameter B ❦

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the reduction is for a variant of LWE where s and e are sampled from a discrete Gaussian distribution of parameter B

Remark: the reduction can be made fully classical [Pei09, BLPRS13]

[Pei09] Peikert. Public-key cryptosystems from the worst-case shortest vector problem. STOC.

[BLPRS13] Brakerski, Langlois, Peikert, Regev, and Stehlé. Classical hardness of learning with errors. STOC

LWE is a lattice problem

LWE instance $\left(\boxed{\boldsymbol{A}}, \boxed{\boldsymbol{b}} = \boxed{\boldsymbol{A}}\boxed{\boldsymbol{s}} + \boxed{\boldsymbol{e}} \bmod q \right)$, $\boldsymbol{e}$ small

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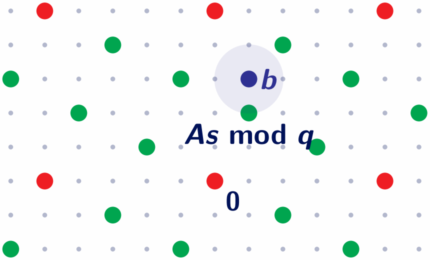
BDD

target $\boldsymbol{b} = \boldsymbol{v} + \boldsymbol{e}$

lattice $\Lambda_q(\boldsymbol{A})$



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BDD

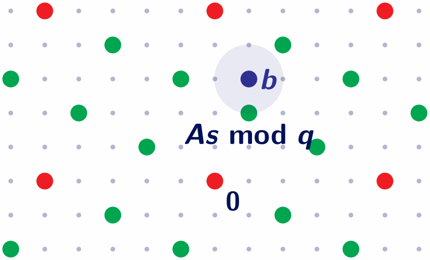
target $\boldsymbol{b} = \boldsymbol{v} + \boldsymbol{e}$

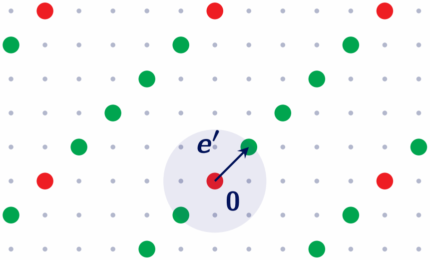
lattice $\boldsymbol{\Lambda}_q(\boldsymbol{A})$

uSVP

lattice $\boldsymbol{\Lambda}_q\left(\begin{pmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{0}_n & \boldsymbol{1} \end{pmatrix}\right)$

contains short $\boldsymbol{e}' := (\boldsymbol{e}^{\perp}, \boldsymbol{1})^{\perp}$





decision-LWE (1)

Sample $\boxed{\boldsymbol{A}} \leftarrow \mathcal{U}(\mathbb{Z}_q^{m\times n})$, $\boxed{\boldsymbol{s}} \leftarrow \mathcal{U}(\mathbb{Z}_q^n)$ and $\boxed{\boldsymbol{e}} \leftarrow \mathcal{U}(\{-B, \cdots, B\}^m)$

Given $\boxed{\boldsymbol{A}}$ and $\boxed{\boldsymbol{b}}$, where $\boxed{\boldsymbol{b}} := \boxed{\boldsymbol{A}}\,\boxed{\boldsymbol{s}} + \boxed{\boldsymbol{e}} \bmod q$ or $\boxed{\boldsymbol{b}} \leftarrow \mathcal{U}(\mathbb{Z}_q^m)$

Guess whether $\boxed{\boldsymbol{b}}$ is uniform or not.

Decision variant of LWE (1)

decision-LWE (1)

Sample $\boxed{\mathbf{A}} \leftarrow \mathcal{U}(\mathbb{Z}_q^{m\times n})$, $\boxed{\mathbf{s}} \leftarrow \mathcal{U}(\mathbb{Z}_q^{n})$ and $\boxed{\mathbf{e}} \leftarrow \mathcal{U}(\{-B,\cdots,B\}^m)$

Given $\boxed{\mathbf{A}}$ and $\boxed{\mathbf{b}}$, where $\boxed{\mathbf{b}} := \boxed{\mathbf{A}}\,\boxed{\mathbf{s}} + \boxed{\mathbf{e}} \bmod q$ or $\boxed{\mathbf{b}} \leftarrow \mathcal{U}(\mathbb{Z}_q^{m})$

Guess whether $\boxed{\mathbf{b}}$ is uniform or not.

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$\Rightarrow$ decision problems can be easier to use for crypto

Decision variant of LWE (2)

if dec-LWE is hard: $\left(\boxed{\boldsymbol{A}}, \boxed{\boldsymbol{b}} = \boxed{\boldsymbol{A}}\,\boxed{\boldsymbol{s}} + \boxed{\boldsymbol{e}} \bmod q\right) \approx \left(\boxed{\boldsymbol{A}}, \boxed{\boldsymbol{b}}\right)$

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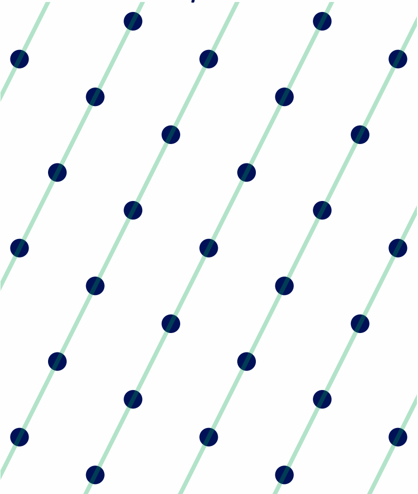
useful in security proofs!

1 bit encryption scheme from LWE

$\Lambda_q^{\perp}(P)$



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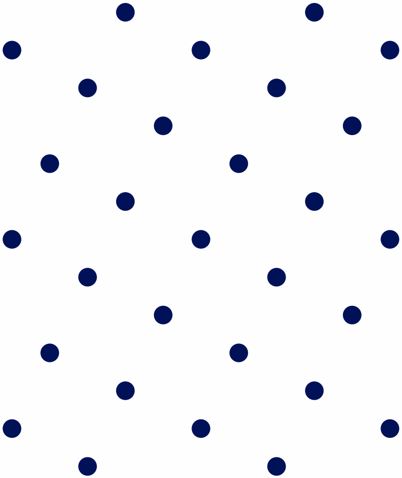

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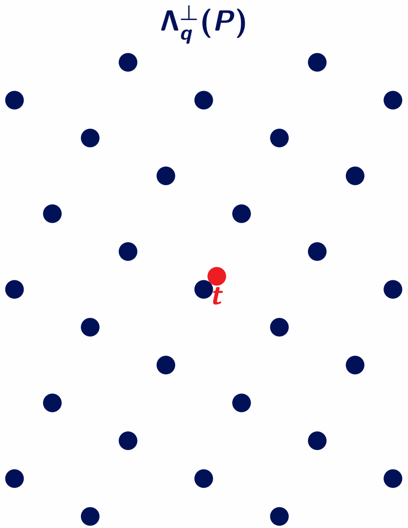

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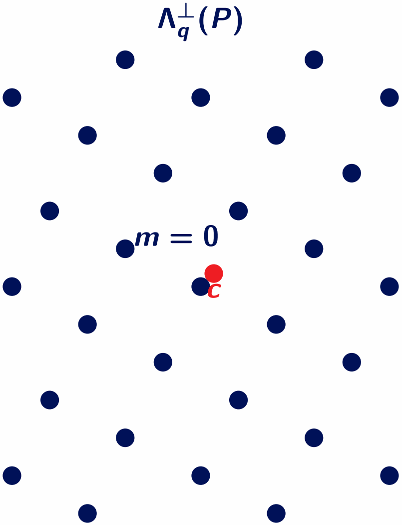

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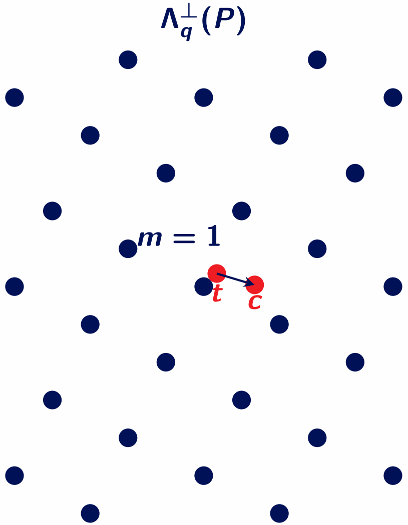

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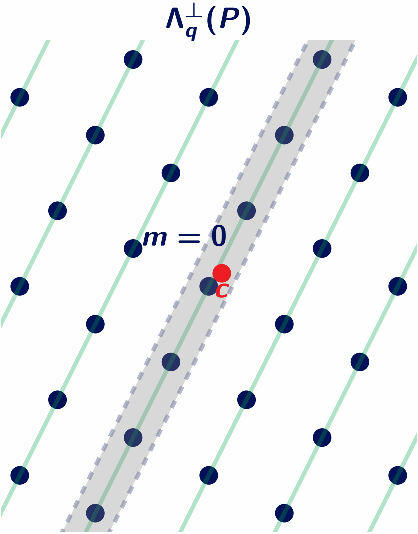

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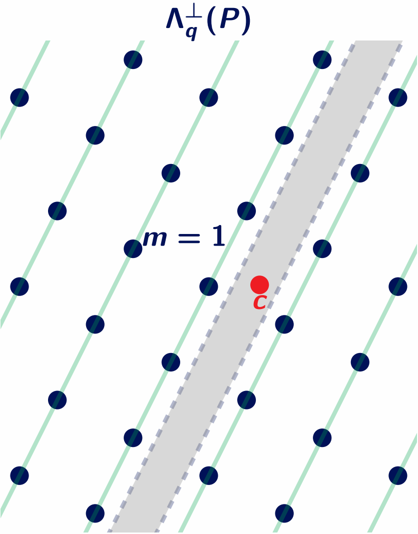

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security proof

dec-LWE $\Rightarrow$

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(no information about sk)

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SIS and LWE are average-case problems

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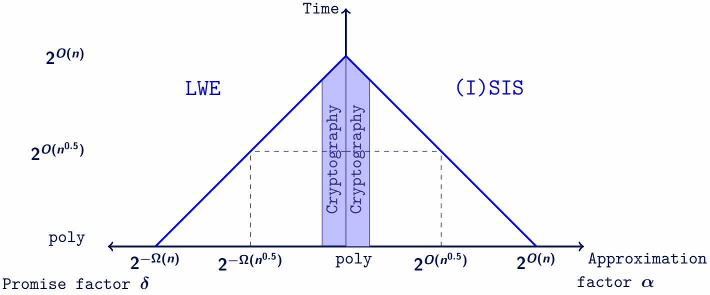
(I)SIS $\overset{\sim}{\longleftrightarrow}$ average-case SVP/CVP

LWE $\overset{\sim}{\longleftrightarrow}$ average case BDD/uSVP

LWE vs SIS



Time
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poly
LWE
(I)SIS
Cryptography
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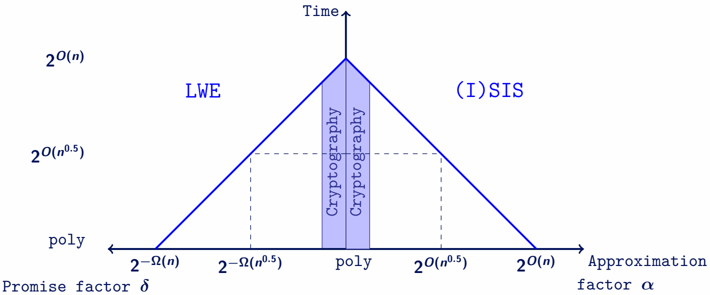

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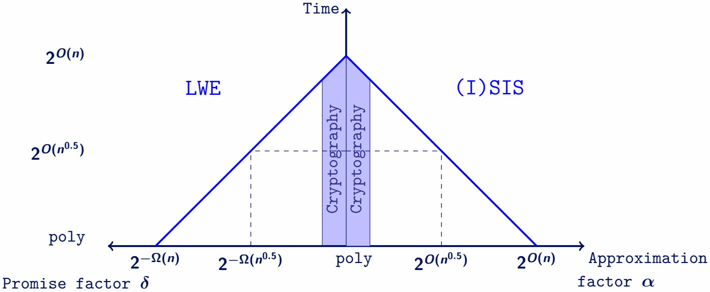

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Exercise

Prove that decision-LWE $\leq$ SIS

Hint: See decryption of LWE encryption scheme

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(decision) LWE / SIS:

- lattice problems over random q-ary lattices

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[Pei16] Peikert. A decade of lattice cryptography. Foundations and trends in theoretical computer science

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Algebraic lattices

Motivation

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Idea: add (algebraic) structure

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Embeddings

$(K = \mathbb{Q}[X]/P(X), \quad \alpha_1, \cdots, \alpha_d$ complex roots of $P(X))$

Coefficient embedding: $\Sigma : \begin{array}{rcl} K & \to & \mathbb{R}^d \\ \sum_{i=0}^{d-1} y_i X^i & \mapsto & (y_0, \cdots, y_{d-1}) \end{array}$

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- for fields used in crypto, both geometries are $\approx$ the same

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Ideal lattices

$\mathcal{O}_K$ is a lattice via the coefficient embedding Σ:

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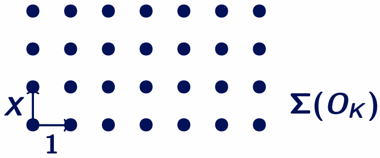
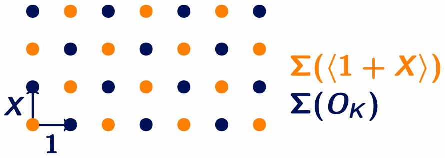
$\Sigma(\langle g \rangle)$ is a lattice of rank d in $\mathbb{Z}^d$ with basis $(\Sigma(g \cdot X^i))_{0 \leq i < d}$

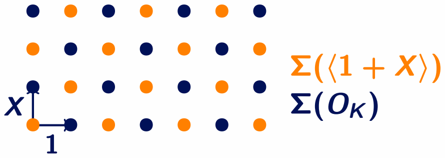
(this is also true for non principal ideals)

(we can replace Σ by σ and $\mathbb{Z}^d$ by $\mathbb{C}^d$)

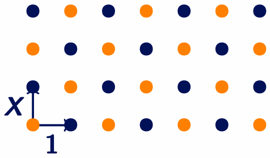
Ideal lattices (2)




X
1
$\Sigma(\langle 1 + X \rangle)$
$\Sigma(\mathcal{O}_K)$



Basis of $\langle g\rangle$: $g, g\cdot X, \cdots, g\cdot X^{d-1}$



$\Sigma(\langle 1+X \rangle)$

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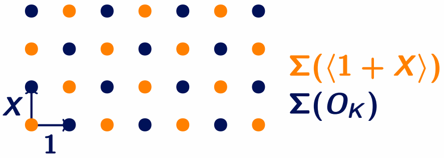
Example in $K = \mathbb{Q}[X]/(X^d + 1$

$$\begin{pmatrix} g_0 & & & \\ g_1 & & & \\ \vdots & & & \\ g_{d-1} & & & \end{pmatrix}$$



We have

$$g \cdot X = \sum_{i=0}^{d-1} g_i X^{i+1} = g_{d-1}X^d + \sum_{i=0}^{d-2} g_i X^{i+1}$$

$$= -g_{d-1} + \sum_{i=0}^{d-2} g_i X^{i+1} \bmod X^d + 1$$

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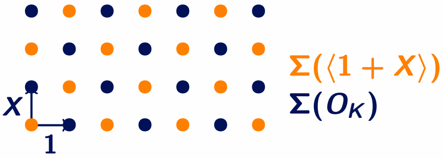

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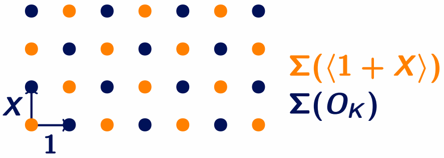

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Storage: n^2 coefficients $\rightarrow$ n

Time: $O(n^2)\rightarrow O(n\log(n))$

(fast polynomial multiplication via FFT)

Module lattices

(Free) module:

$$\boxed{M = \{B \cdot x \mid x \in \mathcal{O}_K^k\} \text{ for some matrix } B \in \mathcal{O}_K^{k \times k} \text{ with } \det_K(B) \neq 0}$$

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Modules vs ideals

Ideal	=	Module of rank 1
(principal ideal	=	free module of rank 1)

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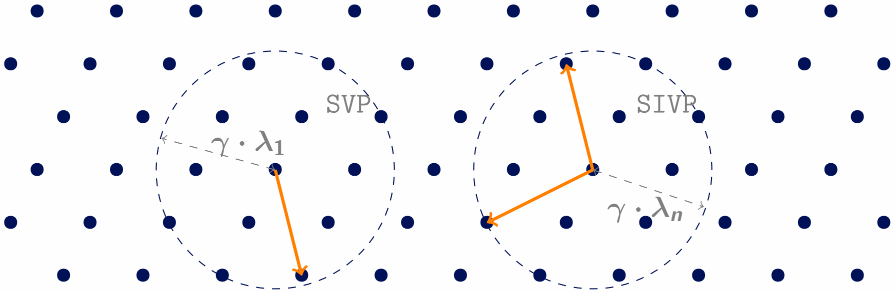
In $K = \mathbb{Q}[X]/(X^d+1)$:

$$M_a = \begin{pmatrix} a_1 & -a_d & \cdots & -a_2 \\ a_2 & a_1 & \cdots & -a_3 \\ \vdots & \ddots & \ddots & \vdots \\ a_d & a_{d-1} & \cdots & a_1 \end{pmatrix}$$

basis of a
principal ideal lattice

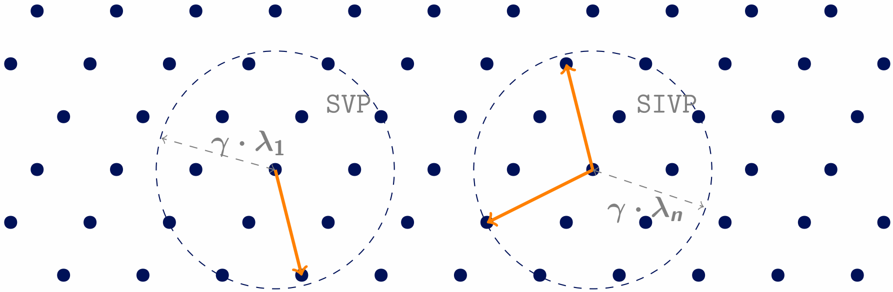
$$d \updownarrow \quad \begin{pmatrix} M_{b_{11}} & M_{b_{12}} & \cdots & M_{b_{1k}} \\ M_{b_{21}} & M_{b_{22}} & \cdots & M_{b_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ M_{b_{k1}} & M_{b_{k2}} & \cdots & M_{b_{kk}} \end{pmatrix} \quad \updownarrow dk$$

basis of a free module lattice
of rank k



γ-SVP
shortest vector problem

γ-SIVP
shortest independent vector problem



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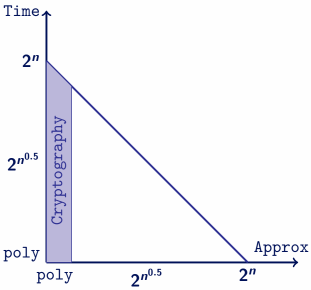
Notations:

- id-X = problem X restricted to ideal lattices
- mod-X$_k$ = problem X restricted to module lattices of rank k

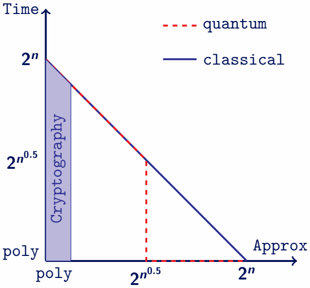
(worst-case: we want algorithms for all ideal/module lattices)

Asymptotics:



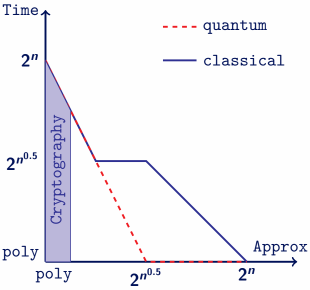
SVP and mod-SVP$_k$

($k \geq 2$)



id-SVP [CDW17]

(in cyclotomic fields)



id-SVP [PHS19,BR20]

(with $2^{O(n)}$ pre-processing)

[CDW17] Cramer, Ducas, Wesolowski. Short stickelberger class relations and application to ideal-SVP. Eurocrypt.

[PHS19] Pellet-Mary, Hanrot, Stehlé. Approx-SVP in ideal lattices with pre-processing. Eurocrypt.

[BR20] Bernard, Roux-Langlois. Twisted-PHS: using the product formula to solve approx-SVP in ideal lattices. AC.

Ring and Module-LWE

(search) mod-LWE$_k$

Parameters: q and B

Problem: Sample

- $A \leftarrow \mathcal{U}((\mathcal{O}_K/q\mathcal{O}_K)^{m\times k})$
- secret $s \in (\mathcal{O}_K/q\mathcal{O}_K)^k$
- error $e \in \mathcal{O}_K^m$ with coefficients in $\{-B, \cdots, B\}$

Given A and $b = A \cdot s + e \bmod q$, recover s

(size of s and e can be defined using Σ or σ)

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$\boxed{\text{RLWE = mod-LWE}_1}$

mod-LWE vs mod-SIVP

$$\text{mod-uSVP}_{m+1} \geq \text{mod-BDD}_m \geq \text{mod-LWE}_k \underset{\text{quantumly!}}{\geq} \text{mod-SIVP}_k$$

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How large should m be?

- as small as possible
- but so that the closest point to $\boldsymbol{b}$ is $\boldsymbol{As}$

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How large should m be?

- as small as possible
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- $m = k$ is not sufficient
- $m = k + 1$ might be sufficient depending on B and q
  - we need roughly $m = k \cdot \frac{\log(q)}{\log(q/B)}$
  - for $k = 1$, $m = 2$ is possible if $B \lesssim \sqrt{q}$

(search) NTRU

Parameters: $q \geq B > 1$

Objective: Sample $f, g \in \mathcal{O}_K$ with coefficients in $\{-B, \cdots, B\}$.
Given $h = f \cdot g^{-1} \bmod q$, recover (f, g)

[HPS98] Hoffstein, Pipher, and Silverman. NTRU: a ring based public key cryptosystem. ANTS.

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dec-NTRU

Parameters: q, B

Objective: distinguish between h as above and h uniform in $\mathcal{O}_K/(q\mathcal{O}_K)$

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NTRU as a lattice

Recall: $h = f \cdot g^{-1} \bmod q$

Definition (NTRU Lattice)

$$\mathcal{L}^{h,q} := \{(a, b) \in R^2 : h \cdot b = a \bmod q\}$$

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$$\text{bad basis} = \begin{pmatrix} q & h \\ 0 & 1 \end{pmatrix}, \quad \text{good basis} = \begin{pmatrix} f & F \\ g & G \end{pmatrix}$$

Two regimes of NTRU

If $\|(f, g)\| \geq \mathrm{poly}(\log n) \cdot \mathrm{gh}(\mathcal{L}^{h,q})$

If $\|(f, g)\| \leq \mathrm{gh}(\mathcal{L}^{h,q})$

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- dec-NTRU is statistically hard

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[SS11] Stehlé and Steinfeld. Making NTRU as secure as worst-case problems over ideal lattices. Eurocrypt.

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uSVP regime $\Rightarrow$ short structured basis

$\Rightarrow$ efficient encryption/signature scheme

(e.g. NTRUEncrypt, NTRUSign, Falcon)

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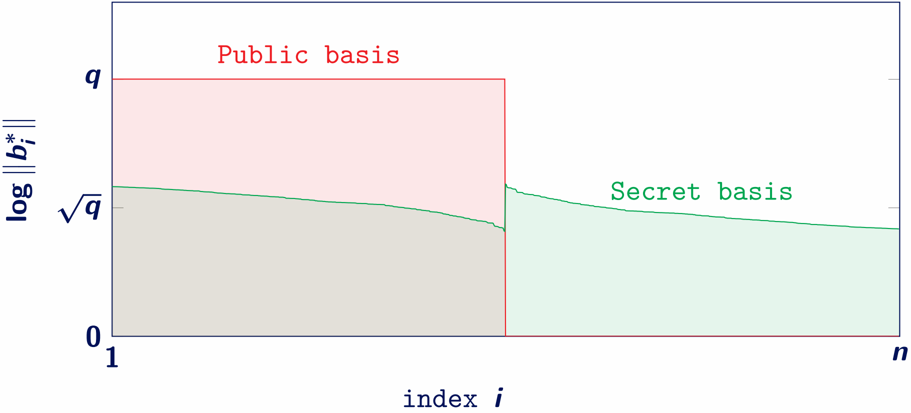
NTRU public vs secret basis

public and secret bases generated from the NTRU problem



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- Most worst-case to average-case reductions still apply
- Ideal lattices $=$ rank 1 modules can be vulnerable
- NIST candidates (e.g. Kyber, Dilithium, Falcon) use rank ≥ 2
  (seems safe so far, but arguably their weakest point)

Conclusion on lattice-based crypto

Advantages:

- many reductions (worst-case to average-case, search to decision, ...)
  - some parameters might still be broken
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